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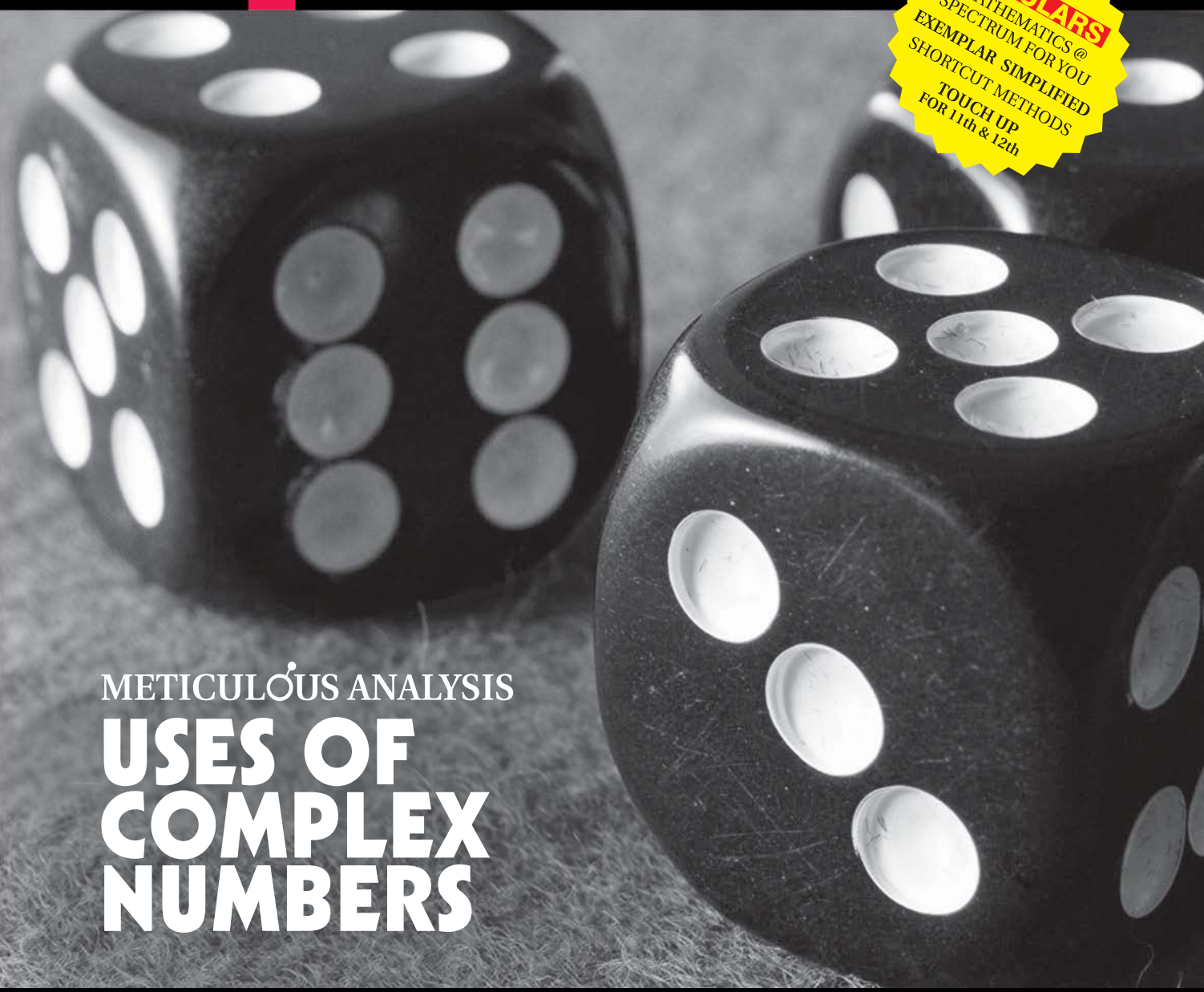

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MATHS GALAXY
RELATION BETWEEN ROOTS AND COEFFICIENTS
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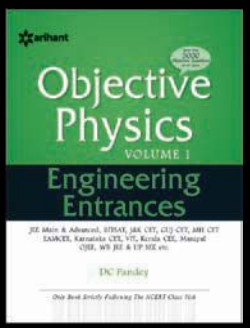

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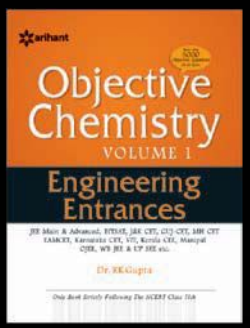

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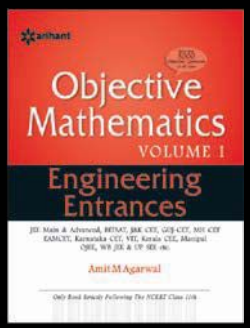

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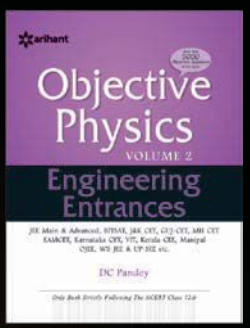

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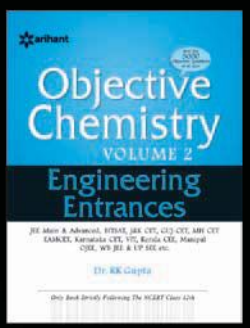

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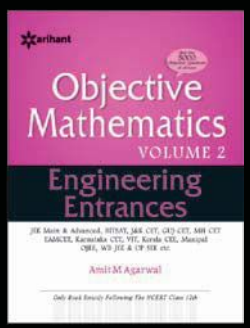

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CONTENTS

RELATIONSHIP BETWEEN ROOTS AND COEFFICIENTS OF *n*TH DEGREE POLYNOMIAL EQUATIONS
4 ---------- MATH GALAXY

TRIGONOMETRIC FUNCTION
12 ..... TOUCH UP XI

FORMULAE AT A GLANCE
16 ..... TRIGONOMETRY

18 ..... JEE MAIN SOLVED PAPER 2016

27 ..... UPSEE SOLVED PAPER 2016

37 ..... SHORTCUT METHODS

40 ..... CONCEPT MAP

SOLUTION OF SYSTEM OF SIMULTANEOUS LINEAR EQUATIONS
42 ..... MATHS GALAXY

RELATION AND FUNCTION
INVERSE TRIGONOMETRIC FUNCTION
51 ..... TOUCH UP XII

SEQUENCES AND SERIES
56 ..... JEE ADVANCED DRILL

63 ..... PERSONALITIES SPECTRUM

64 ..... SPECTRUM FOR YOU

COMPLEX NUMBERS IN COORDINATE GEOMETRY
66 ..... METICULOUS ANALYSIS

75 ..... EXEMPLAR SIMPLIFIED

79 ..... QUIZZER

FROM
THE EDITOR'S DESK

"AT THE END OF THE DAY, THE MOST OVERWHELMING KEY TO A CHILD'S SUCCESS IS THE POSITIVE INVOLVEMENT OF PARENTS."

JANE D. HULL

THE FINEST GUIDE

The role of Parents is very important in a child's education, especially for an IIT aspirant. IIT JEE is the test of both Intelligence and emotional quotient. That's why I consider, parental guidance as important as a mentor's role in sharpening your skills.

Parents play a very important role in the lives of their children. They have to build a solid foundation for their kids in order to provide them a successful and rewarding life. The foundation of an individual can be broken down into several different parts, such as morals, religious beliefs, political standpoints, and education, to name a few. Likewise a child may also learn from many different sources e.g., from teachers at school, from TV programs, from their parents at home etc.

It is said that children learn the first and the most from their parents. I agree with this belief. Children spend most of their time with their parents and they are always watching what they did and how they did it. In most of the cases the first idols of a child's life are their parents. Even though they don´t realize this.

There is another belief that people when became old are going to be just as their parents were. I can see this in my father, who is gradually adapting to my grandfather every day. Even he has started following those things from his father which bothered him earlier. These believes clearly indicate the importance of parents in someone's life.

The moral support that a parent can provide on your failures can't be expected from anybody else. However, sometimes it also

happens that parental support often turns out to be an added pressure. We can define this pressure as **"the pressure of expectation"** or the **"pressure of faith"** in you. This faith may inculcate a fear i.e., the fear of not standing up to their expectations or the fear of failure on the D-day. At such a juncture it is important not to lose focus on the present preparation and self-motivate yourself. Following 5 mantras each for students and parents will help a lot to avoid such situations.

FIVE MANTRAS FOR STUDENTS

1. Talk regularly to your parents. Many aspirants find refuge from parental pressure by reducing communication with them, but instead one must indulge in regular interactions. This will help a lot.
2. Never lied to them about anything. Even tell them you're below par performances and try to accept your failures as your motivation and inspiration. One of my student's was in the habit of sticking all his failures around his study table. According to him these failures always motivate him to do better. Today he is a successful Engineer in Armed Forces.
3. Try to listen them carefully and at the same time you may put your views also but in assertive manner only.
4. **There is the possibility that the views of your parents and you may not match.** In such a case the acceptability at both the ends is necessary e.g., I cannot impose my opinion about chemistry on my son. He may have a different perception about this subject according to his strength or weaknesses. So as a parent it's my duty to listen and understand his perception about this subject too.
5. **Most of your views may contradict with your parents.** This may be due to generation gap or anything else. Try to be open with your parents. Discuss your problems openly to seek their guidance and support.

FIVE MANTRAS FOR PARENTS

1. The support of parents can create wonders for anybody. It is really important for parents not to make their child realise that IIT JEE is the end of the world. Such a realisation creates a mental pressure in student's mind, which very soon becomes a metal block. Formation of this block is tough to overpower and creates a great hindrance in their studies.
2. Secondly, do not impose too much of restrictions on your child. This may direct them towards revolt or non-cordial relations. These non-cordial relations do not holds good and create stress, frustration and finally reflect upon the academic performance along with mental and social health of the student. In such a situation instead of scolding and screaming out in frustration, both sides should discuss their differences with open minds.
3. In my opinion **parents should recommend but not command**; after all, one cannot clear IIT JEE until and unless he/she is self-motivated towards the goal. All that required is a balance between **'parental guidance'** and **'parental pressure'**.
4. While I say that parents should avoid unnecessary interference, it is also important for them to maintain regular touch with their ward, keep records of all tests and ask for explanations for each performance. It is widely accepted that most parents cannot directly help in the studies of their ward but their moral support is very important. At an age when the child is most vulnerable to get misled, parental support is the only thing that can keep things on the right track.
5. Try to develop an ignorance factor as well for your ward. Mix this wisely with your check or restrictions in controlling your ward. Such a mixing always creates positive impact of your actions on your ward's mind.

Remember!

No relation can be as strong as a 'parent-child' relation. It's only your parents and teachers who selflessly wish for your best. Understand their motive of constantly pushing you to study, enquiring about your status in school and coaching, imposing 'restrictions' upon you or any other action on their part which supposedly 'irritates' you. Develop a mutual understanding with them which will help you not just in clearing IIT JEE but also in your future ventures.

Further, always keep following great lines of **ANNE FRANK** *in your mind*

"Parents can only give good advice or show the right path, but the final forming of a person's character lies in their own hands"

Love Aggarwal

RELATIONSHIP BETWEEN ROOTS AND COEFFICIENTS OF *n*TH DEGREE POLYNOMIAL EQUATION

This section of 'theory of equation' is most important for various engineering entrance exams. Every year, atleast one question is asked from this section. This include the relationship between roots and coefficients of a quadratic equation, cubic equation, etc. In theory of equation, this is the most prominent topic from which questions are frequently asked in JEE Main.

Relation between Roots and Coefficients

If $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are the roots of an equation $f(x) \equiv a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0$, then $f(x) = a_0(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_n)$

$\therefore \quad a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n$
$= a_0(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$

On comparing the coefficients of like powers of x on both sides, we get

$$S_1 = \alpha_1 + \alpha_2 + \ldots + \alpha_n = \Sigma\alpha_i = -\frac{a_1}{a_0}$$
$$= \frac{-\text{Coefficient of } x^{n-1}}{\text{Coefficient of } x^n}$$

$$S_2 = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \ldots = \sum_{i \neq j} \alpha_i\alpha_j = (-1)^2\frac{a_2}{a_0}$$
$$= \frac{(-1)^2 \text{ Coefficient of } x^{n-2}}{\text{Coefficient of } x^n}$$

$$S_3 = \alpha_1\alpha_2\alpha_3 + \alpha_2\alpha_3\alpha_4 + \ldots = \sum_{i\neq j\neq k} \alpha_i\alpha_j\alpha_k$$
$$= (-1)^3\frac{a_3}{a_0}$$
$$= \frac{(-1)^3 \text{ Coefficient of } x^{n-3}}{\text{Coefficient of } x^n}$$

$\vdots \quad \vdots$

$$S_n = \alpha_1\alpha_2\alpha_3 \ldots \alpha_n$$
$$= (-1)^n\frac{a_n}{a_0} = \frac{(-1)^n \text{ Constant term}}{\text{Coefficient of } x^n}$$

Particular Cases

Quadratic Equation

If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$.

Example 1 If the sum of the roots of the equation $(a+1)x^2 + (2a+3)x + (3a+4) = 0$ is -1, then the product of the roots is

(a) 1 *(b)* 2 *(c)* 3 *(d)* $a + 1$

Sol. ***(b)*** Let α, β be the roots of the equation $(a+1)x^2 + (2a+3)x + (3a+4) = 0$.

Then, $\alpha + \beta = -1$

$\Rightarrow \quad -\left(\frac{2a+3}{a+1}\right) = -1 \Rightarrow a = -2$

Now, product of the roots $= (3a+4)/(a+1)$
$= (-6+4)/(-2+1)$
$= 2$

SPEED BOOSTER

(i) $a^2 + b^2 = (a+b)^2 - 2ab$

(ii) $(a-b)^2 = (a+b)^2 - 4ab$

(iii) $\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}$

(iv) $\frac{1}{a^2} + \frac{1}{b^2} = \frac{(a+b)^2 - 2ab}{(ab)^2}$

(v) $a^3 + b^3 = (a+b)^3 - 3ab(a+b)$

(vi) $\frac{1}{a^3} + \frac{1}{b^3} = \frac{(a+b)^3 - 3ab(a+b)}{(ab)^3}$

(vii) $a^4 + b^4 = [(a+b)^2 - 2ab]^2 - 2(ab)^2$

Cubic Equation

If α, β and γ are the roots of a cubic equation $ax^3 + bx^2 + cx + d = 0$, then

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} \text{ and } \alpha\beta\gamma = -\frac{d}{a}$$

Example 2 If α, β and γ are the roots of the equation $x^3 + 4x + 1 = 0$, then the value of

$$(\alpha+\beta)^{-1} + (\beta+\gamma)^{-1} + (\gamma+\alpha)^{-1} \text{ is}$$

(a) 2 *(b)* -2 *(c)* 4 *(d)* -4

Sol. **PROBLEM SOLVING STRATEGY** Firstly, find the sum, sum of product of two and product of zero's then use the result to find the required value.

(c) From the equation, we have

$$\alpha + \beta + \gamma = 0,$$
$$\alpha\beta + \beta\gamma + \gamma\alpha = 4 \text{ and } \alpha\beta\gamma = -1$$

Now, $(\alpha+\beta)^{-1} + (\beta+\gamma)^{-1} + (\gamma+\alpha)^{-1}$
$= (-\gamma)^{-1} + (-\alpha)^{-1} + (-\beta)^{-1}$
$= -\frac{1}{\gamma} - \frac{1}{\alpha} - \frac{1}{\beta} = -\left(\frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma}\right) = -\frac{4}{(-1)} = 4$

SPEED BOOSTER

(i) $(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

(ii) $a^3 + b^3 + c^3 - 3abc = 0$, if $a + b + c = 0$

Biquadratic Equation

If α, β, γ and δ are the roots of a biquadratic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$.

Then, $$\alpha + \beta + \gamma + \delta = -\frac{b}{a}$$

$$\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \beta\delta = \frac{c}{a}$$

$$\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta = -\frac{d}{a} \text{ and } \alpha\beta\gamma\delta = \frac{e}{a}$$

Example 3 In equation $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$, if two of its roots are equal in magnitude but opposite in sign, then its roots are

(a) $\pm\sqrt{3}$ (b) $1 \pm i\sqrt{6}$
(c) Both (a) and (b) (d) None of these

Sol. **(c)** Given that $\alpha + \beta = 0$ but $\alpha + \beta + \gamma + \delta = 2$. Hence, $\gamma + \delta = 2$. Let $\alpha\beta = p$ and $\gamma\delta = q$. Therefore, given equation is equivalent to $(x^2 + p)(x^2 - 2x + q) = 0$. Comparing the coefficients, we get $p + q = 4, -2p = 6, pq = -21$. Therefore, $p = -3, q = 7$ and they satisfy $pq = -21$.
Hence, $(x^2 - 3)(x^2 - 2x + 7) = 0$.
$\therefore$ The roots are $\pm\sqrt{3}$ and $1 \pm i\sqrt{6}$.

Formation of a Polynomial Equation

If $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are the roots of nth degree equation, then the equation is given by

$$x^n - S_1x^{n-1} + S_2x^{n-2} - S_3x^{n-3} + \ldots + (-1)^n S_n = 0$$

where, S_i's denotes the sum of the products of the roots taken i at a time.

Particular Cases

Quadratic Equation

If α and β are the roots of a quadratic equation, then the equation is $x^2 - S_1x + S_2 = 0 \Rightarrow x^2 - (\alpha + \beta)x + \alpha\beta = 0$

Example 4 If $\alpha \neq \beta$, $\alpha^2 = 5\alpha - 3$ and $\beta^2 = 5\beta - 3$, then the equation whose roots are α/β and β/α, is

(a) $3x^2 - 19x + 3 = 0$ (b) $19x^2 - 3x + 9 = 0$
(c) $3x^2 + 19x + 3 = 0$ (d) $19x^2 + 3x - 9 = 0$

Sol. PROBLEM SOLVING STRATEGY Firstly, find the sum and product of given equation. Then, using them find the sum and product of roots of new equation.
(a) We have, $\alpha^2 = 5\alpha - 3$ and $\beta^2 = 5\beta - 3$.
Since, α and β are the roots of $x^2 = 5x - 3$
i.e. $x^2 - 5x + 3 = 0$.
$\therefore$ $\alpha + \beta = 5$ and $\alpha\beta = 3$

Now, $S = \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}$

$= \frac{(5)^2 - 2(3)}{3} = \frac{19}{3}$ and $P = \frac{\alpha}{\beta} \cdot \frac{\beta}{\alpha} = 1$

So, the required equation is

$x^2 - Sx + P = 0$

$\Rightarrow x^2 - \frac{19}{3}x + 1 = 0$

$\Rightarrow 3x^2 - 19x + 3 = 0$

Cubic Equation

If α, β and γ are the roots of a cubic equation, then the equation is $x^3 - S_1x^2 + S_2x - S_3 = 0$

$\Rightarrow x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma = 0$

Example 5 If the roots of the equation $x^3 + Px^2 + Qx - 19 = 0$ are each one more than the roots of the equation $x^3 - Ax^2 + Bx - C = 0$, where A, B, C, P and Q are constants, then the value of $A + B + C$ is

(a) -1 (b) 0 (c) 18 (d) 2

Sol. **(c)** Let α, β and γ be the roots of $x^3 - Ax^2 + Bx - C = 0$.
Let $\alpha + 1 = x$ or $\alpha = x - 1$
Therefore, equation having roots $(\alpha + 1), (\beta + 1), (\gamma + 1)$ is
$(x + 1)^3 + P(x + 1)^2 + Q(x + 1) - 19 = 0$
or $x^3 + (3 + P)x^2 + (3 + 2P + Q)x + P + Q - 18 = 0$
According to the question, this equation is same as
$x^3 - Ax^2 + Bx - C = 0$
$\therefore$ $A = -3 - P, B = 3 + 2P + Q, C = 18 - P - Q$
$\therefore$ $A + B + C = 18$

Biquadratic Equation

If α, β, γ and δ are the roots of a biquadratic equation, then the equation is $x^4 - S_1x^3 + S_2x^2 - S_3x + S_4 = 0$

$\Rightarrow x^4 - (\alpha + \beta + \gamma + \delta)x^3 + (\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \beta\delta)x^2$
$- (\alpha\beta\gamma + \alpha\beta\delta + \beta\gamma\delta + \gamma\delta\alpha)x + \alpha\beta\gamma\delta = 0.$

Example 6 If a biquadratic equation has sum of zeros as -2, sum of product of zeros taken 2 at a time is 0, sum of product of zeros taken 3 at a time is 1 and product of zeros is 7, then the equation is

(a) $x^4 + 2x^3 - x + 7 = 0$ (b) $x^4 - x^3 + 2x^2 + 7 = 0$
(c) $x^4 + 2x^3 - x + 10 = 0$ (d) $x^4 + 7x^3 - x - 2 = 0$

Sol. **(a)** We have, $\Sigma\alpha = -2$, $\Sigma\alpha\beta = 0$, $\Sigma\alpha\beta\gamma = 1$ and $\alpha\beta\gamma\delta = 7$.
$\therefore$ The required equation is
$x^4 - (\Sigma\alpha)x^3 + (\Sigma\alpha\beta)x^2 - (\Sigma\alpha\beta\gamma)x + \alpha\beta\gamma\delta = 0$
$\Rightarrow x^4 - (-2)x^3 + (0)x^2 - (1)x + 7 = 0$
$\Rightarrow x^4 + 2x^3 - x + 7 = 0$

SPEED BOOSTER

If the roots of the equation $ax^2 + bx + c = 0$ are α and β, then the equation whose roots are

(i) reciprocal, i.e. $\frac{1}{\alpha}, \frac{1}{\beta}$, is obtained by replacing x by $\frac{1}{x}$.
(ii) negative, i.e $-\alpha, -\beta$ is obtained by replacing x by $-x$.
(iii) squares, i.e. α^2, β^2 is obtained by replacing x by $\sqrt{x}$.
(iv) cubes, i.e. α^3, β^3 is obtained by replacing x by $(x)^{1/3}$.
(v) $\lambda(\neq 0)$ times, is obtained by replacing x by $\frac{x}{\lambda}$.
(vi) λ more than, i.e. $(\alpha + \lambda)$ and $(\beta + \lambda)$, is obtained by replacing x by $(x - \lambda)$.

Smart Practice

Single Option Correct Type

1. If the roots of the quadratic equation $x^2 + px + q = 0$ are $\tan 30°$ and $\tan 15°$ respectively, then the value of $2 + q - p$ is
(a) 1 (b) 2 (c) 3 (d) 4

2. If α, β and γ are the roots of $x^3 - x^2 - 1 = 0$, then the value of $(1+\alpha)/(1-\alpha) + (1+\beta)/(1-\beta) + (1+\gamma)/(1-\gamma)$ is
(a) -5 (b) -6 (c) -7 (d) -2

3. If α and β are the roots of the equation $2x^2 - 35x + 2 = 0$, then the value of $(2\alpha - 35)^3 (2\beta - 35)^3$ is
(a) 8 (b) 16 (c) 64 (d) 128

4. If $\sin\theta$ and $\cos\theta$ are the roots of $ax^2 + bx + c = 0$, then $b^2 - a^2 - 2ac$ is equal to
(a) 0 (b) -1
(c) $\sin\theta\cos\theta$ (d) $\sin\theta + \cos\theta$

5. If the harmonic mean between roots of $(5+\sqrt{2})x^2 - bx + 8 + 2\sqrt{5} = 0$, is 4, then the value of b is
(a) 4 (b) $\sqrt{5}$
(c) $4\sqrt{5}$ (d) $4 + \sqrt{5}$

6. If α and β are the roots of the equation $2x^2 - 3x - 6 = 0$, then the equation whose roots are $\alpha^2 + 2$ and $\beta^2 + 2$, is
(a) $4x^2 + 49x + 118 = 0$ (b) $4x^2 - 49x + 118 = 0$
(c) $4x^2 - 49x - 118 = 0$ (d) $4x^2 - 49x + 18 = 0$

7. If one root of $x^2 - x - k = 0$ is square of the other, then k equals
(a) $2 \pm \sqrt{5}$ (b) $2 \pm \sqrt{3}$
(c) $3 \pm \sqrt{2}$ (d) $5 \pm \sqrt{2}$

8. If α, β are the non-zeros roots of $ax^2 + bx + c = 0$ and α^2, β^2 are the roots of $a^2x^2 + b^2x + c^2 = 0$, then a, b, c are in
(a) GP (b) HP
(c) AP (d) None of these

9. If $\tan\theta_1$, $\tan\theta_2$, $\tan\theta_3$ are the real roots of $x^3 - (a+1)x^2 + (b-a)x - b = 0$, where $\theta_1 + \theta_2 + \theta_3 \in (0, \pi)$, then $\theta_1 + \theta_2 + \theta_3$ is equal to
(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{3\pi}{4}$ (d) π

10. If α and β are the roots of the equation $x^2 + px - 1/(2p^2) = 0$, where $p \in R$. Then, the minimum value of $\alpha^4 + \beta^4$ is
(a) $2\sqrt{2}$ (b) $2 - \sqrt{2}$
(c) 2 (d) $2 + \sqrt{2}$

More than One Options Correct Type

11. If the quadratic equation $ax^2 + bx + c = 0$ has $\sec^2\theta$ and $\text{cosec}^2\theta$ as its roots, then which of the following must hold good?
I. $b + c = 0$ II. $b^2 - 4ac \geq 0$
III. $c \geq 4a$ IV. $4a + b \geq 0$
(a) Only I (b) I, II (c) II, III, IV (d) All of these

12. If the roots of the equation $x^3 + px^2 + qx - 1 = 0$ form an increasing GP, where p and q are real, then
(a) $p + q = 0$, $p \in (3, \infty)$
(b) $p \in (-\infty, -3)$, $p + q = 0$
(c) one of the root is unity, $p + q \neq 0$
(d) one root is smaller than 1 and one root is greater than 1

13. If a, b, c are in GP, then the roots of the equation $ax^2 + bx + c = 0$ are in the ratio
(a) $\frac{1}{2}(-1 + i\sqrt{3})$ (b) $\frac{1}{2}(1 - i\sqrt{3})$
(c) $\frac{1}{2}(-1 \pm i\sqrt{3})$ (d) $\frac{1}{2}(1 + i\sqrt{3})$

14. If α, γ are roots of the equation $Ax^2 - 4x + 1 = 0$ and β, δ are the roots of the equation $Bx^2 - 6x + 1 = 0$, such that α, β, γ and δ are in HP, then
(a) $A = 3, B = 8$ (b) $A = 4, B = 2$ (c) $A = 2, B = 6$ (d) $A = 3, B = 6$

15. If the equation whose roots are the squares of the roots of the cubic $x^3 - ax^2 + bx - 1 = 0$, is identical with the given cubic equation, then
I. $a = 0, b = 3$ II. $a = b = 0$
III. $a = b = 3$ IV. a, b are the roots of $x^2 + x + 2 = 0$
(a) Only I is correct (b) Only I is incorrect
(c) All are correct (d) All are incorrect

16. The real numbers x_1, x_2 and x_3 satisfying the equation $x^3 - x^2 + \beta x + \gamma = 0$ are in AP. Then, all possible values of β are
(a) $\left(-\infty, \frac{1}{3}\right)$ (b) $\left(-\infty, -\frac{1}{3}\right)$ (c) $\left(\frac{1}{3}, \infty\right)$ (d) $\left(-\frac{1}{3}, \infty\right)$

Single Integer Answer Correct Type

17. Suppose a, b, c are the roots of the cubic equation $x^3 - x^2 - 2 = 0$. Then, the value of $a^3 + b^3 + c^3$ is
(a) -7 (b) 7 (c) 49 (d) -49

18. If the roots of the cubic equation $x^3 + ax^2 + bx + c = 0$ are three consecutive positive integers, then the value of $[a^2/(b+1)]$ is
(a) 1 (b) 2 (c) 3 (d) 4

19. a, b and c are all different and non-zero real numbers in AP. If the roots of quadratic equation $ax^2 + bx + c = 0$ are α and β such that $1/\alpha + 1/\beta$, $\alpha + \beta$ and $\alpha^2 + \beta^2$ are in GP, then
(a) $a = 3c$ (b) $a = c$ (c) $c = 3a$ (d) $a = 2c$

20. Let α and β be the roots of the quadratic equation $x^2 - 1154x + 1 = 0$, then the value of $\sqrt[4]{\alpha} + \sqrt[4]{\beta}$ is
(a) 1 *(b)* 4 *(c)* 6 *(d)* 10

21. The function $f(x) = ax^3 + bx^2 + cx + d$ has three positive roots. If the sum of the roots of $f(x)$ is 4, then the largest possible integral value of c/a is
(a) 2 *(b)* 3 *(c)* 4 *(d)* 5

22. **Statement I** If $f(x)$ is a quadratic polynomial satisfying $f(2) + f(4) = 0$. If unit is a root of $f(x) = 0$, then the other root is 3.5.

Statement II If $g(x) \equiv px^2 + qx + r = 0$ has roots α, β, then $\alpha + \beta = -q/p$ and $\alpha\beta = r/p$.
(a) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(b) Statement I is true, Statement II is false
(c) Statement I is false, Statement II is true
(d) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I

23. Let a, b, c, p, q be real numbers. Suppose α, β are the roots of the equation $x^2 + 2px + q = 0$ and $\alpha, 1/\beta$ are the roots of the equation $ax^2 + 2bx + c = 0$, where $\beta \neq 1$.

Statement I $(p^2 - q)(b^2 - ac) \geq 0$

Statement II $b \neq pa$ or $c \neq qa$
(a) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(b) Statement I is true, Statement II is false
(c) Statement I is false, Statement II is true
(d) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I

Match the Columns Type

24. Match the value of column I with the values of column II.

Column I	Column II
A. If the sum of two roots of the equation $x^3 - px^2 + qx - r = 0$ is zero, then	p. $\beta = \frac{r}{q}$
B. If the roots α, β, γ of the equation $x^3 - 3px^2 + 3qx - r = 0$ are in HP, then	q. $\frac{-d}{a}$
C. If the roots of the equation $x^3 - px^2 + qx - r = 0$ are in AP, then	r. $pq = r$
D. If $x^2 + x + 1$ is a factor of $ax^3 + bx^2 + cx + d$, then the real root of $ax^3 + bx^2 + cx + d = 0$ is	s. $2p^3 - 9pq + 27r = 0$

Codes

	A	B	C	D
(a)	p	q	r	s
(b)	r	p	s	q
(c)	q	p	r	s
(d)	s	r	p	q

25. If b is the harmonic mean of a and c and α, β are the roots of the equation
$a(b-c)x^2 + b(c-a)x + c(a-b) = 0$, then
(a) $\alpha + \beta = 3$ *(b)* $\alpha + \beta = \frac{1}{2}$
(c) $\alpha\beta = 2$ *(d)* $\alpha = 1, \beta = 1$

Answers with Explanation

1. ***(c)*** Since, equation $x^2 + px + q = 0$ has roots $\tan 30°$ and $\tan 15°$.

$\therefore \quad \tan 30° + \tan 15° = -p$

and $\quad \tan 30° \tan 15° = q$

Now, $\quad \tan 45° = \tan(30° + 15°)$

$$\Rightarrow \quad 1 = \frac{\tan 30° + \tan 15°}{1 - \tan 30° \tan 15°} \Rightarrow 1 = \frac{-p}{1-q}$$

$$\Rightarrow \quad 1 - q = -p \Rightarrow q - p = 1$$

$$\Rightarrow \quad 2 + q - p = 2 + 1 = 3$$

2. ***(a)*** $\Sigma\alpha = 1, \Sigma\alpha\beta = 0, \alpha\beta\gamma = 1$

$$\Sigma\left(\frac{1+\alpha}{1-\alpha}\right) = -\Sigma\left[\frac{-\alpha + 1 - 2}{1-\alpha}\right] = \Sigma\left[\frac{2}{1-\alpha} - 1\right]$$

$$= 2\,\Sigma\frac{1}{1-\alpha} - 3$$

Now, $\quad \frac{1}{x-\alpha} + \frac{1}{x-\beta} + \frac{1}{x-\gamma} = \frac{3x^2 - 2x}{x^3 - x^2 - 1}$

$$\Rightarrow \quad \frac{1}{1-\alpha} + \frac{1}{1-\beta} + \frac{1}{1-\gamma} = \frac{3-2}{1-1-1} = -1 \quad [\text{put } x = 1]$$

$$\therefore \quad \Sigma\frac{1+\alpha}{1-\alpha} = 2(-1) - 3$$

$$= -2 - 3 = -5$$

3. ***(c)*** Since, α and β are the roots of the equation, we have

$$2x^2 - 35x + 2 = 0$$

$$\therefore \quad 2\alpha^2 - 35\alpha = -2 \Rightarrow 2\alpha - 35 = -\frac{2}{\alpha}$$

and $\quad 2\beta^2 - 35\beta = -2 \Rightarrow 2\beta - 35 = -\frac{2}{\beta}$

$$\text{Now, } (2\alpha - 35)^3 (2\beta - 35)^3 = \left(-\frac{2}{\alpha}\right)^3 \left(-\frac{2}{\beta}\right)^3$$

$$= \frac{8 \times 8}{\alpha^3\beta^3} = \frac{64}{(\alpha\beta)^3} = \frac{64}{(1)^3} \quad [\because \alpha\beta = 1]$$

$$= 64$$

4. ***(a)*** We have, $\sin\theta + \cos\theta = -\frac{b}{a}$ and $\sin\theta\cos\theta = \frac{c}{a}$

Now, we know that

$$\sin^2\theta + \cos^2\theta = 1$$

$$\Rightarrow (\sin\theta + \cos\theta)^2 - 2\sin\theta\cos\theta = 1$$

$$\Rightarrow \quad \left(-\frac{b}{a}\right)^2 - 2\times\frac{c}{a} = 1 \;\Rightarrow\; \frac{b^2}{a^2} - \frac{2c}{a} = 1$$

$$\Rightarrow \quad b^2 - 2ac = a^2$$

$$\Rightarrow \quad b^2 - 2ac - a^2 = 0$$

5. ***(d)*** Let α, β be the roots of the given equation whose HM is 4.

Then, $4 = \dfrac{2\alpha\beta}{\alpha+\beta} \Rightarrow 4 = 2\times\dfrac{\left(\dfrac{8+2\sqrt{5}}{5+\sqrt{2}}\right)}{\left(\dfrac{b}{5+\sqrt{2}}\right)}$ $\left[\because \quad \alpha\beta = \dfrac{8+2\sqrt{5}}{5+\sqrt{2}}, \quad \alpha+\beta = \dfrac{b}{5+\sqrt{2}}\right]$

$$\Rightarrow \quad 2 = \frac{8+2\sqrt{5}}{b} \quad\Rightarrow\quad b = 4+\sqrt{5}$$

6. ***(b)*** Since, α, β are roots of the equation $2x^2 - 3x - 6 = 0$,

we have $\alpha+\beta = \dfrac{3}{2}$ and $\alpha\beta = -3$

$$\Rightarrow \quad \alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = \frac{9}{4} + 6 = \frac{33}{4}$$

Now, $(\alpha^2+2)+(\beta^2+2) = \alpha^2+\beta^2+4 = \dfrac{33}{4}+4 = \dfrac{49}{4}$

and $\quad (\alpha^2+2)(\beta^2+2) = \alpha^2\beta^2 + 2(\alpha^2+\beta^2) + 4$

$$= (3)^2 + 2\left(\frac{33}{4}\right) + 4 = \frac{59}{2}$$

So, the equation whose roots are α^2+2 and β^2+2, is given by

$$x^2 - x[(\alpha^2+2)+(\beta^2+2)] + (\alpha^2+2)(\beta^2+2) = 0$$

$$\Rightarrow \quad x^2 - \frac{49}{4}x + \frac{59}{2} = 0 \;\Rightarrow 4x^2 - 49x + 118 = 0$$

7. ***(a)*** Let α and α^2 be the roots of $x^2 - x - k = 0$.

Then, $\quad \alpha + \alpha^2 = 1$ and $\alpha^3 = -k$

$$\Rightarrow \quad (-k)^{1/3} + (-k)^{2/3} = 1$$

$$\Rightarrow \quad -k^{1/3} + k^{2/3} = 1$$

$$\Rightarrow \quad (k^{2/3} - k^{1/3})^3 = 1$$

$$\Rightarrow \quad k^2 - k - 3k\,(k^{2/3} - k^{1/3}) = 1$$

$$\Rightarrow \quad k^2 - k - 3k(1) = 1$$

$$\Rightarrow \quad k^2 - 4k - 1 = 0$$

$$\Rightarrow \quad k = 2 \pm \sqrt{5}$$

8. ***(a)*** Since, α, β are roots of $ax^2 + bx + c = 0$.

$$\therefore \quad \alpha+\beta = -\frac{b}{a} \qquad \ldots(i)$$

and $\quad \alpha\beta = \dfrac{c}{a} \qquad \ldots(ii)$

Also, α^2, β^2 are the roots of $a^2x^2 + b^2x + c^2 = 0$.

$$\therefore \quad \alpha^2+\beta^2 = -\frac{b^2}{a^2} \qquad \ldots(iii)$$

and $\quad \alpha^2\beta^2 = \dfrac{c^2}{a^2} \qquad \ldots(iv)$

Now, from Eq. (iii), we get

$$(\alpha+\beta)^2 - 2\alpha\beta = -\frac{b^2}{a^2}$$

$$\Rightarrow \quad \left(-\frac{b}{a}\right)^2 - 2\frac{c}{a} = \frac{-b^2}{a^2} \Rightarrow \frac{2b^2}{a^2} = \frac{2c}{a}$$

$$\Rightarrow \quad b^2 = ac$$

$\therefore$ a, b, c are in GP.

9. ***(b)*** $\tan\theta_1 + \tan\theta_2 + \tan\theta_3 = (a+1)$

$$\Sigma\tan\theta_1\tan\theta_2 = (b-a)$$

$$\tan\theta_1\tan\theta_2\tan\theta_3 = b$$

$$\therefore \quad \tan(\theta_1+\theta_2+\theta_3) = \frac{\Sigma\tan\theta_1 - \Pi\tan\theta_1}{1-\Sigma\tan\theta_1\tan\theta_2}$$

$$= \frac{a+1-b}{1-(b-a)} = \frac{a-b+1}{a-b+1}$$

$$= 1$$

$$\Rightarrow \quad \theta_1+\theta_2+\theta_3 = \frac{\pi}{4}$$

10. ***(d)*** $\alpha^4 + \beta^4 = (\alpha^2+\beta^2)^2 - 2\alpha^2\beta^2$

$$= [(\alpha+\beta)^2 - 2\alpha\beta]^2 - 2(\alpha\beta)^2$$

$$= \left(p^2 + \frac{1}{p^2}\right)^2 - \frac{1}{2p^4}$$

$$= p^4 + \frac{1}{2p^4} + 2$$

$$= \left(p^2 - \frac{1}{\sqrt{2}p^2}\right)^2 + 2 + \sqrt{2} \geq 2 + \sqrt{2}$$

Thus, the minimum value of $\alpha^4 + \beta^4$ is $2+\sqrt{2}$.

11. ***(d)*** We know that,

$$\sec^2\theta + \operatorname{cosec}^2\theta = \sec^2\theta\cdot\operatorname{cosec}^2\theta$$

$$\Rightarrow \quad -\frac{b}{a} = \frac{c}{a}$$

$$\Rightarrow \quad b + c = 0$$

Also, $\quad b^2 - 4ac \geq 0$

or $\quad c^2 - 4ac \geq 0$

$$\Rightarrow \quad c\,(c-4a) \geq 0$$

$$\Rightarrow \quad c - 4a \geq 0$$

Further, $\quad b^2 + 4ab \geq 0$

$$\Rightarrow \quad b\,(b+4a) \geq 0$$

$$\Rightarrow \quad b + 4a \geq 0$$

12. ***(d)*** Let the roots be $\dfrac{a}{r}, a, ar$

where, $a > 0, r > 1$.

Now, $\quad \dfrac{a}{r} + a + ar = -p \qquad \ldots(i)$

$$\frac{a}{r}\cdot a + a\cdot(ar) + ar\left(\frac{a}{r}\right) = q \qquad \ldots(ii)$$

and $\quad \dfrac{a}{r}\cdot a\cdot ar = 1 \qquad \ldots(iii)$

$$\Rightarrow \quad a^3 = 1$$

$$\Rightarrow \quad a = 1$$

Put $a = 1$ in Eq. (i), we get

$$\frac{1}{r} + 1 + r = -p \qquad \ldots(iv)$$

$$r + \frac{1}{r} = -p - 1$$

$$\Rightarrow \quad -p - 1 > 2 \qquad \left[\because r + \frac{1}{r} > 2\right]$$

$$\Rightarrow \quad p < -3$$

From Eq. (ii), we get

$$\frac{1}{r} + r + 1 = q \qquad \ldots(v)$$

From Eqs. (iv) and (v), we get

$$-p = q \Rightarrow p + q = 0$$

Now, as $r > 1$

$\therefore$ $\frac{a}{r} = \frac{1}{r} < 1$ and $ar = r > 1$

13. ***(c)*** Given, $b^2 = ac$

$\Rightarrow$ $\left(\frac{b}{a}\right)^2 = \frac{c}{a}$

$\Rightarrow$ $\left(-\frac{b}{a}\right)^2 = \frac{c}{a}$

$\Rightarrow$ $(\alpha + \beta)^2 = \alpha\beta$

$\Rightarrow$ $\alpha^2 + \beta^2 + 2\alpha\beta = \alpha\beta$

$\Rightarrow$ $\alpha^2 + \beta^2 + \alpha\beta = 0$

$\Rightarrow$ $\left(\frac{\alpha}{\beta}\right)^2 + \left(\frac{\alpha}{\beta}\right) + 1 = 0$

$\Rightarrow$ $\frac{\alpha}{\beta} = \frac{-1 \pm \sqrt{3}i}{2}$

14. ***(a)*** Since, $\alpha, \beta, \gamma, \delta$ are in HP, then $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}$ are in AP and they may be taken as $a - 3d, a - d, a + d, a + 3d$. Replacing x by $\frac{1}{x}$, we get the equation whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}$. Therefore, equation $x^2 - 4x + A = 0$ has roots $a - 3d$, $a + d$ and equation $x^2 - 6x + B = 0$ has roots $a - d, a + 3d$. Sum of the roots is

$$2(a - d) = 4, 2(a + d) = 6$$

$\therefore$ $a = \frac{5}{2}, d = \frac{1}{2}$

Product of the roots is $(a - 3d)(a + d) = A = 3$

$$(a - d)(a + 3d) = B = 8$$

15. ***(b)*** Given equation is $x^3 - ax^2 + bx - 1 = 0$.

If roots of the equation are α, β, γ, then

$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$= a^2 - 2b$

$\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma)$

$= b^2 - 2a$

$\alpha^2\beta^2\gamma^2 = 1$

So, the equation whose roots are $\alpha^2, \beta^2, \gamma^2$ is given by

$$x^3 - (a^2 - 2b)x^2 + (b^2 - 2a)x - 1 = 0$$

It is identical to

$$x^3 - ax^2 + bx - 1 = 0$$

$\Rightarrow$ $a^2 - 2b = a$ and $b^2 - 2a = b$

Eliminating b, we get

$$\frac{(a^2 - a)^2}{4} - 2a = \frac{a^2 - a}{2}$$

$\Rightarrow$ $a\{a(a - 1)^2 - 8 - 2(a - 1)\} = 0$

$\Rightarrow$ $a\{a^3 - 2a^2 - a - 6\} = 0$

$\Rightarrow$ $a(a - 3)(a^2 + a + 2) = 0$

$\Rightarrow$ $a = 0$ or $a = 3$ or $a^2 + a + 2 = 0$

which gives $b = 0$ or $b = 3$ or $b^2 + b + 2 = 0$.

So, $a = b = 0$ or $a = b = 3$ or a, b are roots of $x^2 + x + 2 = 0$.

16. ***(a)*** From the question, the real roots of $x^3 - x^2 + \beta x + \gamma = 0$ are x_1, x_2, x_3 and they are in AP. As x_1, x_2, x_3 are in AP.

Let $x_1 = a - d, x_2 = a, x_3 = a + d$

Now, $x_1 + x_2 + x_3 = -\frac{(-1)}{1} = 1$

$\Rightarrow$ $a - d + a + a + d = 1 \Rightarrow a = \frac{1}{3}$ ...(i)

$x_1x_2 + x_2x_3 + x_3x_1 = \frac{\beta}{1} = \beta$

$\Rightarrow$ $(a - d)a + a(a + d) + (a + d)(a - d) = \beta$ ...(ii)

$x_1x_2x_3 = -\frac{\gamma}{1} = -\gamma$

$\Rightarrow$ $(a - d)a(a + d) = -\gamma$ ...(iii)

From Eqs. (i) and (ii), we get

$$3a^2 - d^2 = \beta$$

$\Rightarrow$ $3\left(\frac{1}{9}\right) - d^2 = \beta \Rightarrow \beta = \frac{1}{3} - d^2 < \frac{1}{3}$

17. ***(b)*** Given, $a + b + c = 1$ ...(i)

$ab + bc + ca = 0$ ...(ii)

$abc = 2$ ...(iii)

Now, $(a + b + c)^2 = 1$

$\Rightarrow$ $a^2 + b^2 + c^2 + 2(ab + bc + ca) = 1$

$\therefore$ $a^2 + b^2 + c^2 = 1$

Now, $a^3 + b^3 + c^3 - 3abc = (a + b + c)(\Sigma a^2 - \Sigma ab)$

$= 1(1 - 0) = 1$

$\Rightarrow$ $a^3 + b^3 + c^3 = 1 + 3abc$

$= 1 + 3 \times 2 = 7$

18. ***(c)*** Let $n, n + 1, n + 2$ be three consecutive positive integers.

$\therefore$ Sum $= 3(n + 1) = -a \Rightarrow 9(n + 1)^2 = a^2$

Sum of the roots taken 2 at a time $= b$

$\therefore$ $n(n + 1) + (n + 1)(n + 2) + (n + 2)n = b$

$\Rightarrow$ $n^2 + n + n^2 + 3n + 2 + n^2 + 2n = b$

$\Rightarrow$ $b = 3n^2 + 6n + 2$

$\Rightarrow$ $b + 1 = 3n^2 + 6n + 3$

$\Rightarrow$ $b + 1 = 3(n + 1)^2$

$\Rightarrow$ $b + 1 = \frac{a^2}{3}$

$\therefore$ $\frac{a^2}{b + 1} = 3$

19. ***(a)*** We have, $\frac{1}{\alpha} + \frac{1}{\beta}, \alpha + \beta, \alpha^2 + \beta^2$ are in GP.

$\Rightarrow$ $(\alpha + \beta)^2 = \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)(\alpha^2 + \beta^2)$

$\Rightarrow$ $(\alpha + \beta)^2 = \frac{\alpha + \beta}{\alpha\beta}[(\alpha + \beta)^2 - 2\alpha\beta]$

$\Rightarrow$ $\frac{b^2}{a^2} = -\frac{b}{c}\left[\frac{b^2}{a^2} - \frac{2c}{a}\right]$ $\left[\because \alpha + \beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a}\right]$

$\Rightarrow$ $cb^2 + b(b^2 - 2ac) = 0$

$\Rightarrow$ $b[bc + b^2 - 2ac] = 0$

but $b \neq 0$

$\therefore$ $bc + b^2 - 2ac = 0$

Since, a, b and c are in AP.

$\therefore \quad b = \dfrac{a+c}{2}$

Therefore, we have $\dfrac{(a+c)c}{2} + \left(\dfrac{a+c}{2}\right)^2 - 2ac = 0$

$\Rightarrow \quad a^2 - 4ac + 3c^2 = 0$

or $\quad (a-c)(a-3c) = 0$

but $\quad a \neq c \Rightarrow a = 3c$

$\Rightarrow \quad \dfrac{a}{c} = 3 \quad \Rightarrow \quad a = 3c$

20. ***(c)*** We have, $\alpha + \beta = 1154$ and $\alpha\beta = 1$

Now, $(\sqrt{\alpha} + \sqrt{\beta})^2 = \alpha + \beta + 2\sqrt{\alpha\beta}$

$= 1154 + 2 = 1156$

$\Rightarrow \quad \sqrt{\alpha} + \sqrt{\beta} = 34$

Again, $(\alpha^{1/4} + \beta^{1/4})^2 = \sqrt{\alpha} + \sqrt{\beta} + 2(\alpha\beta)^{1/4}$

$= 34 + 2 = 36$

$\Rightarrow \quad \alpha^{1/4} + \beta^{1/4} = 6$

21. ***(d)*** Let $ax^3 + bx^2 + cx + d = 0$ has roots p, q, r.

Then, $\quad pq + qr + rp = \dfrac{c}{a} \quad ...(i)$

But $\quad pq + qr + rp \leq p^2 + q^2 + r^2$

$= (p + q + r)^2 - 2\Sigma pq$

$\therefore \quad 3(pq + qr + rp) \leq (p + q + r)^2$

$\Rightarrow \quad 3(pq + qr + rp) \leq (4)^2$

$\Rightarrow \quad pq + qr + rp \leq \dfrac{16}{3} \Rightarrow \dfrac{c}{a} \leq \dfrac{16}{3}$

$\therefore$ Largest possible integral value of $\dfrac{c}{a}$ is 5.

22. ***(a)*** $\quad f(x) = (x-1)(ax+b)$

$f(2) = 2a + b$

$f(4) = 3(4a + b) = 12a + 3b$

$\because \quad f(2) + f(4) = 0$

$\Rightarrow \quad 2a + b + 12a + 3b = 0$

$\Rightarrow \quad 14a + 4b = 0$

$\Rightarrow \quad -\dfrac{b}{a} = 3.5$

Now, sum of roots $= \dfrac{a-b}{a} = 1 - \dfrac{b}{a} = 1 - (-3.5) = 4.5$

Hence, the other root is 3.5.

23. ***(a)*** Since, α, β are the roots of the equation $x^2 + 2px + q = 0$ and $\alpha, \dfrac{1}{\beta}$ are the roots of the equation $ax^2 + 2bx + c = 0$.

$\alpha + \beta = -2p, \alpha\beta = q, \alpha + \dfrac{1}{\beta} = -\dfrac{2b}{a}, \dfrac{\alpha}{\beta} = \dfrac{c}{a}$

$$\therefore (p^2 - q)(b^2 - ac) = \left\{\left(\frac{\alpha+\beta}{-2}\right)^2 - \alpha\beta\right\}\left\{\left(\frac{\alpha+\frac{1}{\beta}}{-2}\right)^2 - \frac{\alpha}{\beta}\right\}a^2$$

$$= \frac{(\alpha-\beta)^2}{4} \times \frac{1}{4} \times \left(\alpha - \frac{1}{\beta}\right)^2 a^2 \geq 0$$

So, Statement 1 is correct.

If $\beta = 1$, then

$\alpha + \beta = -2p \Rightarrow \alpha + 1 = -2p$

and $\quad \alpha + \dfrac{1}{\beta} = \dfrac{-2b}{a} \quad \Rightarrow \quad \alpha + 1 = -\dfrac{2b}{a}$

$\therefore \quad -2p = -\dfrac{2b}{a} \Rightarrow b = pa$

But $\beta \neq 1$ (given). Therefore, $b \neq pa$.

Also, $\beta = 1$

$\Rightarrow \quad \alpha = q$ and $\alpha = \dfrac{c}{a} \quad \Rightarrow \quad q = \dfrac{c}{a}$ or $c = aq$

But $\beta \neq 1$. So, $c \neq qa$.

So, Statement II is true. But it is not a correct explanation for Statement I.

24. ***(b)***

A. Let the roots of the given equation be α, β, γ such that

$\alpha + \beta = 0$.

Then, $\quad \alpha + \beta + \gamma = \dfrac{-(-p)}{1} \Rightarrow \alpha + \beta + \gamma = p$

$\Rightarrow \quad \gamma = p$

$\therefore \quad \gamma^3 - p\gamma^2 + q\gamma - r = 0$

$\Rightarrow \quad p^3 - p^3 + pq - r = 0$

$\Rightarrow \quad pq = r$

B. Clearly, $\dfrac{1}{\alpha}, \dfrac{1}{\beta}, \dfrac{1}{\gamma}$ are the roots of the equation

$-rx^3 + 3qx^2 - 3px + 1 = 0$ and are in AP.

Now, $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} = \dfrac{3q}{r}$

$\Rightarrow \quad \dfrac{3}{\beta} = \dfrac{3q}{r} \qquad \left[\because \dfrac{1}{\alpha} + \dfrac{1}{\gamma} = \dfrac{2}{\beta}\right]$

$\Rightarrow \quad \beta = \dfrac{r}{q}$

C. Let the roots of the given equation be $a - d, a, a + d$.

Then, $(a-d) + a + (a+d) = \dfrac{-(-p)}{1} \Rightarrow a = \dfrac{p}{3}$

Since, a is a root of the given equation.

$\therefore \quad a^3 - pa^2 + qa - r = 0 \quad \Rightarrow \quad \dfrac{p^3}{27} - \dfrac{p^3}{9} + \dfrac{qp}{3} - r = 0$

$\Rightarrow \quad 2p^3 - 9pq + 27r = 0$

D. Since, $x^2 + x + 1$ is the factor.

$\therefore \omega, \omega^2$ are the roots of $ax^3 + bx^2 + cx + d = 0$

$\therefore \omega \cdot \omega^2 \cdot \alpha = -\dfrac{d}{a} \quad \Rightarrow \quad \alpha = -\dfrac{d}{a}$

25. ***(d)*** We observe that the sum of the coefficients of the given equation is zero. Therefore, 1 is a root of the given equation.

Let $\alpha = 1$ and β be the other root.

Then, $\quad \beta \times 1 = \dfrac{c(a-b)}{a(b-c)} = \dfrac{c}{a}\left\{\dfrac{a - \dfrac{2ac}{a+c}}{\dfrac{2ac}{a+c} - c}\right\}$

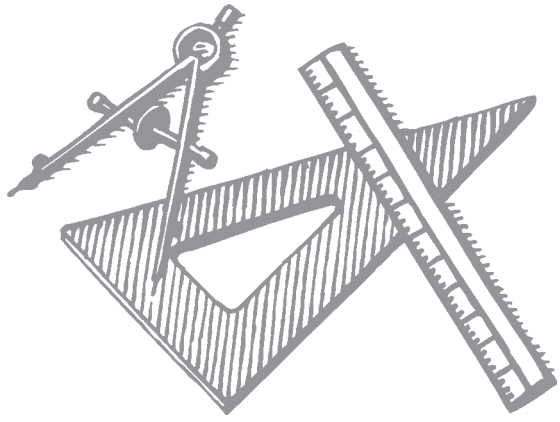
$= \dfrac{c}{a}\left\{\dfrac{a^2 + ac - 2ac}{2ac - ac - c^2}\right\}$

$= \dfrac{c}{a}\left[\dfrac{a^2 - ac}{ac - c^2}\right] = \dfrac{c}{a} \cdot \dfrac{a(a-c)}{c(a-c)} = 1$

Hence, $\quad \alpha = \beta = 1$

TOUCH UP

TRIGONOMETRIC FUNCTION FOR CLASS XI



Topicwise Collection of best subjective problems

Very Short Answer Type Questions [1 Mark]

1. Prove that $\tan 75° + \cot 75° = 4$.

2. If $2\tan\beta + \cot\beta = \tan\alpha$, then prove that $\cot\beta = 2\tan(\alpha - \beta)$.

3. Show that $\sqrt{2+\sqrt{2+\sqrt{2+2\cos 8\theta}}} = 2\cos\theta, 0 < \theta < \frac{\pi}{8}$.

4. Evaluate $108\sin\frac{\pi}{18} - 144\sin^3\frac{\pi}{18}$.

5. Prove that $\sin^2 24° - \sin^2 6° = \frac{\sqrt{5}-1}{8}$

6. If $\cos 4x = 1 + k\sin^2 x\cos^2 x$, then write the value of k.

7. In any ΔABC, prove that $\Delta = \frac{b^2 + c^2 - a^2}{4\cot A}$.

8. If $\sin 2A = \lambda \sin 2B$, then write the value of $\frac{\lambda+1}{\lambda-1}$.

9. If $\alpha + \beta = 90°$, then find the maximum and minimum values of $\sin\alpha\sin\beta$.

10. Prove that $\sec^2\theta + \operatorname{cosec}^2\theta \geq 4$.

Short Answer Type Questions [4 Marks]

11. Find the value of the expression

$$3\left\{\sin^4\left(\frac{3\pi}{2} - \theta\right) + \sin^4(3\pi + \theta)\right\} - 2\left\{\sin^6\left(\frac{\pi}{2} + \theta\right) + \sin^6(5\pi - \theta)\right\}.$$

12. If $\tan A - \tan B = x$ and $\cot B - \cot A = y$, then prove that $\cot(A - B) = \frac{1}{x} + \frac{1}{y}$.

13. If $\sin(y + z - x), \sin(z + x - y), \sin(x + y - z)$ are in AP, then prove that $\tan x, \tan y, \tan z$ are also in AP.

14. Prove that

$$\sin^4\frac{\pi}{8} + \sin^4\frac{3\pi}{8} + \sin^4\frac{5\pi}{8} + \sin^4\frac{7\pi}{8} = \frac{3}{2}.$$

15. Prove that $\cot\frac{\pi}{24} = \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{6}$.

16. Prove that $\frac{\tan 3x}{\tan x}$ never lies between $\frac{1}{3}$ and 3.

Long Answer Type Questions [6 Marks]

17. If $A = \cos^2\theta + \sin^4\theta$, then prove that $\frac{3}{4} \leq A \leq 1$ for all values of θ.

18. If α and β are the solutions of $a\cos\theta + b\sin\theta = c$, then show that

(i) $\cos(\alpha + \beta) = \frac{a^2 - b^2}{a^2 + b^2}$

(ii) $\cos(\alpha - \beta) = \frac{2c^2 - (a^2 + b^2)}{a^2 + b^2}$

19. The angle of elevation of the top of a tower from a point A due South of the tower is α and from B due East of the tower is β. If $AB = d$, then show that the height of the tower is

$$\frac{d}{\sqrt{\cot^2\alpha + \cot^2\beta}}.$$

20. With usual notations, if in a ΔABC, $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$, then prove that $\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$.

Answers with Explanation

1. We have,

$$\tan 75^\circ = \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

$$\therefore \quad \cot 75^\circ = \frac{1}{\tan 75^\circ} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

Now, $\text{LHS} = \tan 75^\circ + \cot 75^\circ$

$$= \frac{\sqrt{3}+1}{\sqrt{3}-1} + \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{(\sqrt{3}+1)^2 + (\sqrt{3}-1)^2}{(\sqrt{3}-1)(\sqrt{3}+1)}$$

$$= \frac{3+1+2\sqrt{3}+3+1-2\sqrt{3}}{3-1}$$

$$= \frac{8}{2} = 4$$ **Hence proved.**

2. Clearly,

$$2\tan(\alpha-\beta) = 2\left\{\frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}\right\}$$

$$= 2\left\{\frac{2\tan\beta + \cot\beta - \tan\beta}{1+(2\tan\beta+\cot\beta)\tan\beta}\right\}$$

$$= 2\left\{\frac{\tan\beta + \cot\beta}{1+2\tan^2\beta+1}\right\} = 2\left\{\frac{\tan\beta + \frac{1}{\tan\beta}}{2+2\tan^2\beta}\right\}$$

$$= \frac{2(\tan^2\beta+1)}{2(1+\tan^2\beta)\tan\beta} = \cot\beta$$ **Hence proved.**

3. We have,

$$\text{LHS} = \sqrt{2+\sqrt{2+\sqrt{2(1+\cos 8\theta)}}}$$

$$= \sqrt{2+\sqrt{2+\sqrt{2(2\cos^2 4\theta)}}}$$

$$= \sqrt{2+\sqrt{2+2\cos 4\theta}} = \sqrt{2+\sqrt{2(1+\cos 4\theta)}}$$

$$= \sqrt{2+\sqrt{2(2\cos^2 2\theta)}} = \sqrt{2+2\cos 2\theta}$$

$$= \sqrt{2(1+\cos 2\theta)} = \sqrt{2(2\cos^2\theta)}$$

$$= 2\cos\theta$$ **Hence proved.**

4. We have,

$$\text{LHS} = 108\sin\frac{\pi}{18} - 144\sin^3\frac{\pi}{18}$$

$$= 36\left(3\sin\frac{\pi}{18} - 4\sin^3\frac{\pi}{18}\right)$$

$$= 36\sin 3\times\frac{\pi}{18} = 36\sin\frac{\pi}{6}$$

$$= 36\times\frac{1}{2} = 18$$

5. We have,

$$\text{LHS} = \sin^2 24^\circ - \sin^2 6^\circ$$

$$= \sin(24^\circ+6^\circ)\sin(24^\circ-6^\circ)$$

$$[\because \sin^2 A - \sin^2 B = \sin(A+B)\sin(A-B)]$$

$$= \sin 30^\circ \sin 18^\circ$$

$$= \frac{1}{2}\times\frac{\sqrt{5}-1}{4}$$

$$= \frac{\sqrt{5}-1}{8}$$ **Hence proved.**

6. We have,

$$\cos 4x = \cos 2(2x)$$

$$= 1-2\sin^2 2x \qquad [\because \cos 2\theta = 1-2\sin^2\theta]$$

$$= 1-2(2\sin x\cos x)^2 \qquad [\because \sin 2\theta = 2\sin\theta\cos\theta]$$

$$= 1-8\sin^2 x\cos^2 x$$

$\therefore\ k = -8$

7. We have,

$$\text{RHS} = \frac{b^2+c^2-a^2}{4\cot A} = \frac{b^2+c^2-a^2}{4\cos A}\cdot\sin A$$

$$= \frac{b^2+c^2-a^2}{4(b^2+c^2-a^2)}\cdot 2bc\sin A$$

$$= \frac{1}{2}bc\sin A = \Delta$$ **Hence proved.**

8. We have,

$$\sin 2A = \lambda\sin 2B \Rightarrow \lambda = \frac{\sin 2A}{\sin 2B}$$

$$\Rightarrow \quad \frac{\lambda}{1} = \frac{\sin 2A}{\sin 2B} \Rightarrow \frac{\lambda+1}{\lambda-1} = \frac{\sin 2A+\sin 2B}{\sin 2A-\sin 2B}$$

[by applying componendo and dividendo]

$$\Rightarrow \quad \frac{\lambda+1}{\lambda-1} = \frac{2\sin(A+B)\cos(A-B)}{2\cos(A+B)\sin(A-B)}$$

$$\Rightarrow \quad \frac{\lambda+1}{\lambda-1} = \frac{\tan(A+B)}{\tan(A-B)}$$

9. Let $y = \sin\alpha\sin\beta$

Then, $y = \frac{1}{2}(2\sin\alpha\sin\beta) = \frac{1}{2}\{\cos(\alpha-\beta) - \cos(\alpha+\beta)\}$

$$= \frac{1}{2}\{\cos(\alpha-\beta) - \cos 90^\circ\} = \frac{1}{2}\cos(\alpha-\beta)$$

We know that,

$$-1 \le \cos(\alpha-\beta) \le 1$$

$$\Rightarrow \quad \frac{-1}{2} \le \frac{1}{2}\cos(\alpha-\beta) \le \frac{1}{2} \Rightarrow \frac{-1}{2} \le y \le \frac{1}{2}$$

$$\Rightarrow \quad \frac{-1}{2} \le \sin\alpha\sin\beta \le \frac{1}{2}$$

Hence, $\frac{-1}{2}$ and $\frac{1}{2}$ are respectively the minimum and maximum values of $\sin\alpha\sin\beta$.

10. We have,

$$\sec^2\theta + \operatorname{cosec}^2\theta = 1+\tan^2\theta+1+\cot^2\theta$$

$$= 2+\tan^2\theta+\cot^2\theta$$

$$= 2+\tan^2\theta+\cot^2\theta$$

$$-2\tan\theta\cot\theta + 2\tan\theta\cot\theta$$

$$= 2+(\tan\theta-\cot\theta)^2+2$$

$$= 4+(\tan\theta-\cot\theta)^2$$

$$\ge 4 \qquad [\because (\tan\theta-\cot\theta)^2 \ge 0]$$

Hence proved.

11. Given,

$$3\left\{\sin^4\left(\frac{3\pi}{2}-\theta\right)+\sin^4(3\pi+\theta)\right\} - 2$$

$$\left\{\sin^6\left(\frac{\pi}{2}+\theta\right)+\sin^6(5\pi-\theta)\right\}$$

$$= 3\{(-\cos\theta)^4 + (-\sin\theta)^4\} - 2\{(\cos\theta)^6 + (\sin\theta)^6\}$$

$= 3(\cos^4\theta + \sin^4\theta) - 2(\cos^6\theta + \sin^6\theta)$
$= 3\{(\cos^2\theta + \sin^2\theta)^2 - 2\sin^2\theta\cos^2\theta\} - 2$
$\{(\cos^2\theta + \sin^2\theta)^3 - 3\cos^2\theta\sin^2\theta$
$(\cos^2\theta + \sin^2\theta)\}$
$= 3(1 - 2\sin^2\theta\cos^2\theta) - 2(1 - 3\cos^2\theta\sin^2\theta)$
$= 3 - 6\sin^2\theta\cos^2\theta - 2 + 6\cos^2\theta\sin^2\theta = 1$

12. We have,

$$\tan A - \tan B = x \text{ and } \cot B - \cot A = y$$

Now, $\cot B - \cot A = y$

$$\Rightarrow \frac{1}{\tan B} - \frac{1}{\tan A} = y \Rightarrow \frac{\tan A - \tan B}{\tan A \tan B} = y$$

$$\Rightarrow \frac{x}{\tan A \tan B} = y \Rightarrow \tan A \tan B = \frac{x}{y}$$

$$\cot(A - B) = \frac{1}{\tan(A - B)}$$
$$= \frac{1 + \tan A \tan B}{\tan A - \tan B}$$
$$= \frac{1 + \frac{x}{y}}{x} = \frac{y + x}{xy}$$
$$= \frac{1}{x} + \frac{1}{y}$$

Hence proved.

13. It is given that $\sin(y + z - x)$, $\sin(z + x - y)$ and $\sin(x + y - z)$ are in AP.

$\therefore \sin(z + x - y) - \sin(y + z - x)$
$= \sin(x + y - z) - \sin(z + x - y)$

$\Rightarrow 2\cos z \sin(x - y) = 2\cos x \sin(y - z)$

$\Rightarrow \sin(x - y)\cos z = \cos x \sin(y - z)$

$\Rightarrow \sin x \cos y \cos z - \cos x \sin y \cos z$
$= \sin y \cos z \cos x - \cos y \sin z \cos x$

$\Rightarrow 2\sin y \cos x \cos z = \sin x \cos y \cos z$
$+ \cos x \cos y \sin z$

$\Rightarrow 2\tan y = \tan x + \tan z$
[dividing throughout by $\cos x \cos y \cos z$]

$\Rightarrow \tan x, \tan y, \tan z$ are in AP. **Hence proved.**

14. We can write that

$$\sin^4\frac{7\pi}{8} = \sin^4\left(\pi - \frac{\pi}{8}\right) = \sin^4\frac{\pi}{8}$$

and $$\sin^4\frac{5\pi}{8} = \sin^4\left(\pi - \frac{3\pi}{8}\right) = \sin^4\frac{3\pi}{8}$$

$$\therefore \text{LHS} = \sin^4\frac{\pi}{8} + \sin^4\frac{3\pi}{8} + \sin^4\frac{5\pi}{8} + \sin^4\frac{7\pi}{8}$$
$$= \sin^4\frac{\pi}{8} + \sin^4\frac{3\pi}{8} + \sin^4\frac{3\pi}{8} + \sin^4\frac{\pi}{8}$$
$$= 2\left[\sin^4\frac{\pi}{8} + \sin^4\frac{3\pi}{8}\right]$$
$$= 2\left\{\left(\sin^2\frac{\pi}{8}\right)^2 + \left(\sin^2\frac{3\pi}{8}\right)^2\right\}$$
$$= 2\left\{\left(\frac{1 - \cos\pi/4}{2}\right)^2 + \left(\frac{1 - \cos 3\pi/4}{2}\right)^2\right\}$$
$$= \frac{2}{4}\left\{\left(1 - \cos\frac{\pi}{4}\right)^2 + \left(1 - \cos\frac{3\pi}{4}\right)^2\right\}$$
$$= \frac{1}{2}\left\{\left(1 - \frac{1}{\sqrt{2}}\right)^2 + \left(1 + \frac{1}{\sqrt{2}}\right)^2\right\}$$
$$= \frac{1}{2}\left\{1 + \frac{1}{2} - \frac{2}{\sqrt{2}} + 1 + \frac{1}{2} + \frac{2}{\sqrt{2}}\right\}$$
$$= \frac{1}{2} \times 3 = \frac{3}{2}$$
$= \text{RHS}$ **Hence proved.**

15. We have,

$$\text{LHS} = \cot\frac{\pi}{24} = \frac{\cos\frac{\pi}{24}}{\sin\frac{\pi}{24}} = \frac{2\cos\frac{\pi}{24}\cos\frac{\pi}{24}}{2\sin\frac{\pi}{24}\cos\frac{\pi}{24}}$$
$$= \frac{2\cos^2\frac{\pi}{24}}{2\sin\frac{\pi}{24}\cos\frac{\pi}{24}} = \frac{1 + \cos\frac{\pi}{12}}{\sin\frac{\pi}{12}}$$
$$= \frac{1 + \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)} = \frac{1 + \frac{\sqrt{3} + 1}{2\sqrt{2}}}{\frac{\sqrt{3} - 1}{2\sqrt{2}}}$$
$$= \frac{2\sqrt{2} + \sqrt{3} + 1}{\sqrt{3} - 1} = \frac{(2\sqrt{2} + \sqrt{3} + 1)(\sqrt{3} + 1)}{(\sqrt{3} - 1)(\sqrt{3} + 1)}$$
$$= \frac{2\sqrt{6} + 2\sqrt{2} + 3 + \sqrt{3} + \sqrt{3} + 1}{3 - 1}$$
$$= \frac{2\sqrt{6} + 2\sqrt{2} + 2\sqrt{3} + 4}{2}$$
$$= \sqrt{2} + \sqrt{3} + 2 + \sqrt{6} = \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{6}$$

Hence proved.

16. Let $y = \frac{\tan 3x}{\tan x}$

Then, $y = \frac{3\tan x - \tan^3 x}{\tan x(1 - 3\tan^2 x)} \Rightarrow y = \frac{3 - \tan^2 x}{1 - 3\tan^2 x}$

$\Rightarrow (3y - 1)\tan^2 x = (y - 3)$

$\Rightarrow \tan^2 x = \frac{y - 3}{3y - 1}$

But $\tan^2 x \geq 0$ for all x,

$\Rightarrow \frac{y - 3}{3y - 1} \geq 0$

$\therefore y < \frac{1}{3}$ or $y \geq 3$

$\Rightarrow y$ does not lie between $\frac{1}{3}$ and 3.

Hence, $\frac{\tan 3x}{\tan x}$ never lies between $\frac{1}{3}$ and 3. **Hence proved.**

17. We have,

$A = \cos^2\theta + \sin^4\theta$
$= \cos^2\theta + (\sin^2\theta)^2$

Now, $-1 \leq \sin\theta \leq 1$ for all θ

$\Rightarrow 0 \leq \sin^2\theta \leq 1$ for all $\theta \Rightarrow (\sin^2\theta)^2 \leq \sin^2\theta$
[for $0 < x < 1$, $x^n < x$ for all $n \in N - \{1\}$]

$\Rightarrow \cos^2\theta + (\sin^2\theta)^2 \le \cos^2\theta + \sin^2\theta$ for all θ

$\Rightarrow A \le 1$ for all θ ...(i)

Again, $A = \cos^2\theta + \sin^4\theta$

$= 1 - \sin^2\theta + (\sin^2\theta)^2$

$= 1 - \frac{1}{4} + \left\{\frac{1}{4} - \sin^2\theta + (\sin^2\theta)^2\right\}$

$= \frac{3}{4} + \left(\frac{1}{2} - \sin^2\theta\right)^2$

Now, $\left(\frac{1}{2} - \sin^2\theta\right)^2 \ge 0$ for all θ

$\Rightarrow \frac{3}{4} + \left(\frac{1}{2} - \sin^2\theta\right)^2 \ge \frac{3}{4}$ for all θ

$\Rightarrow A \ge \frac{3}{4}$ for all θ ...(ii)

From Eqs. (i) and (ii), we get

$\frac{3}{4} \le A \le 1$ for all θ. **Hence proved.**

18. We have,

$a\cos\theta + b\sin\theta = c$ ...(i)

$\Rightarrow a\cos\theta = c - b\sin\theta$

$\Rightarrow a^2\cos^2\theta = (c - b\sin\theta)^2$

$\Rightarrow a^2(1 - \sin^2\theta) = c^2 - 2bc\sin\theta + b^2\sin^2\theta$

$\Rightarrow (a^2 + b^2)\sin^2\theta - 2bc\sin\theta + (c^2 - a^2) = 0$ ...(ii)

Since, α, β are roots of Eq. (i). Therefore, $\sin\alpha$ and $\sin\beta$ are roots of Eq. (ii).

$$\sin\alpha\sin\beta = \frac{c^2 - a^2}{a^2 + b^2} \quad \text{...(iii)}$$

Again, $a\cos\theta + b\sin\theta = c$

$\Rightarrow b\sin\theta = c - a\cos\theta$

$\Rightarrow b^2\sin^2\theta = c^2 - 2ac\cos\theta + a^2\cos^2\theta$

$\Rightarrow b^2(1 - \cos^2\theta) = c^2 - 2ac\cos\theta + a^2\cos^2\theta$

$\Rightarrow (a^2 + b^2)\cos^2\theta - 2ac\cos\theta + c^2 - b^2 = 0$ ...(iv)

It is given that α, β are the roots of Eq. (i). So, $\cos\alpha, \cos\beta$ are the roots of Eq. (iv).

$\therefore \cos\alpha\cos\beta = \frac{c^2 - b^2}{a^2 + b^2}$

(i) $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= \frac{c^2 - b^2}{a^2 + b^2} - \frac{c^2 - a^2}{a^2 + b^2}$

$= \frac{a^2 - b^2}{a^2 + b^2}$

(ii) $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

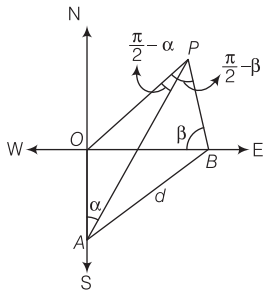
$= \frac{c^2 - b^2}{a^2 + b^2} + \frac{c^2 - a^2}{a^2 + b^2}$

$= \frac{2c^2 - (a^2 + b^2)}{a^2 + b^2}$ **Hence proved.**

19. Let OP be the tower and let A and B be two points due South and East respectively of the tower such that

$\angle OAP = \alpha$ and $\angle OBP = \beta$.

Then, $\angle OPA = \frac{\pi}{2} - \alpha$ and $\angle OPB = \frac{\pi}{2} - \beta$



Using sine rule in ΔOAP and ΔOBP, we have

$$\frac{OA}{\sin\left(\frac{\pi}{2} - \alpha\right)} = \frac{OP}{\sin\alpha}$$

and $$\frac{OB}{\sin\left(\frac{\pi}{2} - \beta\right)} = \frac{OP}{\sin\beta}$$

$\Rightarrow \frac{OA}{\cos\alpha} = \frac{OP}{\sin\alpha}$ and $\frac{OB}{\cos\beta} = \frac{OP}{\sin\beta}$

$\Rightarrow OA = OP\cot\alpha$ and $OB = OP\cot\beta$

Now, in ΔAOB,

$AB^2 = OA^2 + OB^2$

$\Rightarrow d^2 = OP^2\cot^2\alpha + OP^2\cot^2\beta$

$\Rightarrow OP = \frac{d}{\sqrt{\cot^2\alpha + \cot^2\beta}}$ **Hence proved.**

20. Let $\frac{b + c}{11} = \frac{c + a}{12} = \frac{a + b}{13} = \lambda$

$\Rightarrow b + c = 11\lambda, c + a = 12\lambda, a + b = 13\lambda$

$\therefore b + c + c + a + a + b = 11\lambda + 12\lambda + 13\lambda$

$\Rightarrow 2(a + b + c) = 36\lambda$

$\Rightarrow a + b + c = 18\lambda$

$\therefore a = 7\lambda, b = 6\lambda, c = 5\lambda$

Now, $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

$= \frac{36\lambda^2 + 25\lambda^2 - 49\lambda^2}{60\lambda^2}$

$= \frac{12}{60} = \frac{1}{5}$

$\cos B = \frac{c^2 + a^2 - b^2}{2ca}$

$= \frac{25\lambda^2 + 49\lambda^2 - 36\lambda^2}{70\lambda^2}$

$= \frac{38}{70} = \frac{19}{35}$

$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

$= \frac{49\lambda^2 + 36\lambda^2 - 25\lambda^2}{84\lambda^2} = \frac{60}{84} = \frac{5}{7}$

$\therefore \cos A : \cos B : \cos C$

$= \frac{1}{5} : \frac{19}{35} : \frac{5}{7} = 7 : 19 : 25$ **Hence proved.**

Trigonometry

1. Trigonometric Functions

Trigonometric Functions	Domain	Range
$\sin x$	R	$[-1, 1]$
$\cos x$	R	$[-1, 1]$
$\tan x$	$R - \left\{(2n+1)\frac{\pi}{2}, n \in I\right\}$	R
$\operatorname{cosec} x$	$R - \{n\pi, n \in I\}$	$R - (-1, 1)$
$\sec x$	$R - \left\{(2n+1)\frac{\pi}{2}, n \in I\right\}$	$R - (-1, 1)$
$\cot x$	$R - \{n\pi, n \in I\}$	R

2. Trigonometric Ratios of Some Special Angles

α	$7\frac{1}{2}^\circ$	15°	$22\frac{1}{2}^\circ$	18°	36°
$\sin\alpha$	$\frac{\sqrt{4-\sqrt{2}-\sqrt{6}}}{2\sqrt{2}}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$
$\cos\alpha$	$\frac{\sqrt{4+\sqrt{2}+\sqrt{6}}}{2\sqrt{2}}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{5}+1}{4}$
$\tan\alpha$	$(\sqrt{3}-\sqrt{2})$ $(\sqrt{2}-1)$	$2-\sqrt{3}$	$\sqrt{2}-1$	$\frac{\sqrt{25-10\sqrt{15}}}{5}$	$\sqrt{5-2\sqrt{5}}$

3. Trigonometric Ratios of Some Useful angles

(i) $\sin 75^\circ = \cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$ (ii) $\cos 75^\circ = \sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$

(iii) $\tan 75^\circ = \cot 15^\circ = 2+\sqrt{3}$ (iv) $\cot 75^\circ = \tan 15^\circ = 2-\sqrt{3}$

(v) $\sin 9^\circ = \sqrt{\frac{1-\cos 18^\circ}{2}} = \frac{\sqrt{3+\sqrt{5}}-\sqrt{5-\sqrt{5}}}{4} = \cos 81^\circ$

(vi) $\cos 9^\circ = \sqrt{\frac{1+\cos 18^\circ}{2}} = \frac{\sqrt{3+\sqrt{5}}+\sqrt{5-\sqrt{5}}}{4} = \sin 81^\circ$

4. Maximum and Minimum Values of Trigonometric Expressions

(i) $-\sqrt{a^2+b^2} \le a\cos\theta \pm b\sin\theta \le \sqrt{a^2+b^2}$

(ii) $c-\sqrt{a^2+b^2} \le a\cos\theta \pm b\sin\theta + c \le c+\sqrt{a^2+b^2}$

5. Trigonometric Ratios of Compound Angles

(i) $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$

(ii) $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

(iii) $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$ (iv) $\cot(A \pm B) = \frac{\cot A\cot B \mp 1}{\cot B \pm \cot A}$

6. (i) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$

(ii) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$

7. Transformation formulae

(i) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

(ii) $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

(iii) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

(iv) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

(v) $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

(vi) $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

(vii) $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

(viii) $\cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

8. Trigonometric Ratios of Multiple Angles

(i) $\sin 2A = 2\sin A\cos A = \frac{2\tan A}{1+\tan^2 A}$

(ii) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1$
$= 1 - 2\sin^2 A = \frac{1-\tan^2 A}{1+\tan^2 A}$

(iii) $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$ (iv) $\sin 3A = 3\sin A - 4\sin^3 A$

(v) $\cos 3A = 4\cos^3 A - 3\cos A$

(vi) $\tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A}$

9. Trigonometric Ratios of Angle $A/2$ in Terms of $\cos A$

(i) $\sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}}$ (ii) $\cos\frac{A}{2} = \sqrt{\frac{1+\cos A}{2}}$

(iii) $\tan\frac{A}{2} = \sqrt{\frac{1-\cos A}{1+\cos A}}$

10. (i) $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin\{\alpha+(n-1)\beta\}$
$= \frac{\sin\left[\alpha+(n-1)\frac{\beta}{2}\right]\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$

(ii) $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos\{\alpha+(n-1)\beta\}$
$= \frac{\cos\left[\alpha+(n-1)\frac{\beta}{2}\right]\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$

11. If A, B, C are angles of a triangle. Then,

(i) $\sin(B+C) = \sin A$ (ii) $\cos(B+C) = -\cos A$

(iii) $\sin\left(\frac{B+C}{2}\right) = \cos\frac{A}{2}$ (iv) $\cos\left(\frac{B+C}{2}\right) = \sin\frac{A}{2}$

(v) $\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$

(vi) $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A\cos B\cos C$

(vii) $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(viii) $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(ix) $\tan A + \tan B + \tan C = \tan A\tan B\tan C$

x) $\cot A\cot B + \cot B\cot C + \cot C\cot A = 1$

(xi) $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

(xii) $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

12. $\cos A\cos 2A\cos 2^2 A \ldots \cos 2^{n-1} A = \frac{1}{2^n \sin A}\sin(2^n A)$

13. (i) $\cos A + \cos B + \cos C$ is maximum, if $A = B = C$

(ii) $\sin A + \sin B + \sin C$ is maximum, if $A = B = C$

(iii) $\tan A + \tan B + \tan C$ is minimum, if $A = B = C$

(iv) $\cot A + \cot B + \cot C$ is minimum, if $A = B = C$.

(v) $\cos^2 A + \cos^2 B + \cos^2 C = 1 \Rightarrow \Delta ABC$ is right angled triangle.

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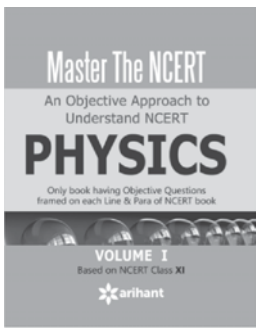
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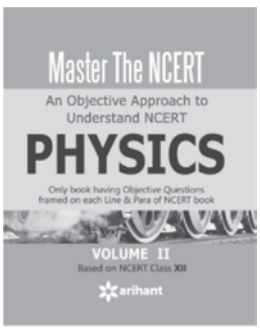
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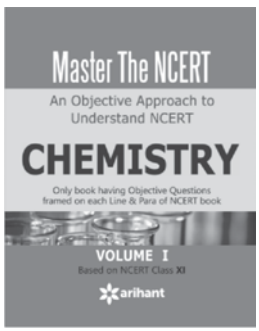
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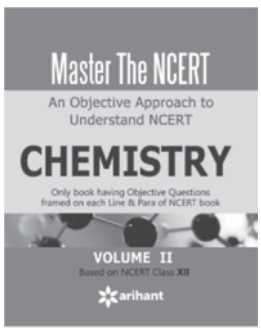
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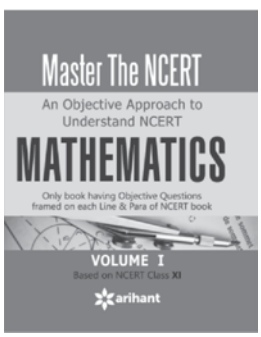
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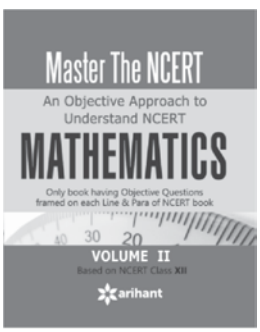
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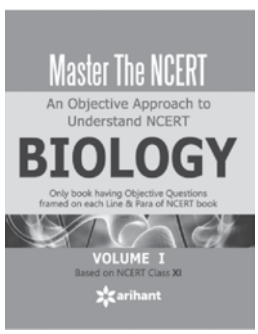
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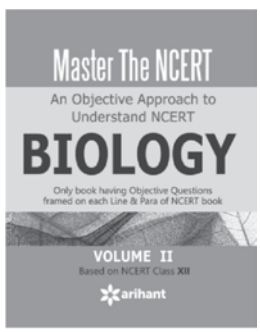
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PAPER I (Online)

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INSTRUCTIONS

- There are three parts in the question paper.
- Each question is allotted 4 marks for correct response.
- Candidates will be awarded marks as stated above for correct response of each question. One-fourth marks will be deducted for indicating incorrect response of each question. No deduction from the total score will be made, if no response is indicated for an item in the answer sheet.
- There is only one correct response for each question. Filling up more than one response in any question will be treated as wrong response and marks for wrong response will be deducted according as per instructions.

EXAM CRUX

This year Mathematics question paper was quite balanced as 30% questions were HOTS, 40% questions were moderate and 30% questions were easy. Overall, it was easier as compared to previous year papers. In classwise division of questions it is found that 50% questions are asked from class-XI and 50% questions are asked from class-XII.

According to branchwise division, it is found that 4 questions are asked from *Coordinate Geometry*, 2 questions are asked from *3D-Geometry*, 10 questions are asked from *Calculus*, 10 questions are asked from *Algebra*, 2 questions are asked from *Trigonometry*, 1 question is asked from *Statistics* and 1 question is asked from *Vector*.

1. For $x \in R$, $x \neq 0$, $x \neq 1$, let $f_0(x) = 1/1-x$ and $f_{n+1}(x) = f_0(f_n(x))$, $n = 0, 1, 2, \ldots$. Then, the value of $f_{100}(3) + f_1\left(\frac{2}{3}\right) + f_2\left(\frac{3}{2}\right)$ is

(a) $\frac{8}{3}$ (b) $\frac{5}{3}$ (c) $\frac{4}{3}$ (d) $\frac{1}{3}$

2. The point represented by $2 + i$ in the argand plane moves 1 unit eastwards, then 2 units northwards and finally from there, $2\sqrt{2}$ units in the South-westwards direction. Then, its new position in the argand plane is at the point represented by

(a) $2 + 2i$ (b) $1 + i$
(c) $-1 - i$ (d) $-2 - 2i$

3. If the equations $x^2 + bx - 1 = 0$ and $x^2 + x + b = 0$ have a common root different from -1, then $|b|$ is equal to

(a) $\sqrt{2}$ (b) 2 (c) 3 (d) $\sqrt{3}$

4. If $P = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $Q = PAP^T$, then $P^T Q^{2015} P$ is equal to

(a) $\begin{bmatrix} 0 & 2015 \\ 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 2015 & 1 \\ 0 & 2015 \end{bmatrix}$
(c) $\begin{bmatrix} 2015 & 0 \\ 1 & 2015 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2015 \\ 0 & 1 \end{bmatrix}$

5. The number of distinct real roots of the equation, $\begin{vmatrix} \cos x & \sin x & \sin x \\ \sin x & \cos x & \sin x \\ \sin x & \sin x & \cos x \end{vmatrix} = 0$ in the interval $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ is

(a) 4 (b) 3 (c) 2 (d) 1

6. If the four letter words (need not be meaningful) are to be formed using the letters from the word "MEDITERRANEAN" such that the first letter is R and the fourth letter is E, then the total number of all such words is

(a) $\frac{11!}{(2!)^3}$ (b) 110 (c) 56 (d) 59

7. For $x \in R$, $x \neq -1$, if
$(1+x)^{2016} + x(1+x)^{2015} + x^2(1+x)^{2014} + \ldots + x^{2016} = \sum_{i=0}^{2016} a_i x^i$, then a_{17} is equal to

(a) $\frac{2017!}{17!\,2000!}$ (b) $\frac{2016!}{17!\,1999!}$ (c) $\frac{2017!}{2000!}$ (d) $\frac{2016!}{16!}$

8. Let x, y, z be positive real numbers such that $x+y+z=12$ and $x^3y^4z^5=(0.1)(600)^3$. Then, $x^3+y^3+z^3$ is equal to

(a) 270 (b) 258 (c) 342 (d) 216

9. The value of $\sum_{r=1}^{15} r^2\left(\frac{{}^{15}C_r}{{}^{15}C_{r-1}}\right)$ is

(a) 560 (b) 680 (c) 1240 (d) 1085

10. If $\lim_{x\to\infty}\left(1+\frac{a}{x}-\frac{4}{x^2}\right)^{2x} = e^3$, then a is equal to

(a) 2 (b) 3/2 (c) 2/3 (d) 1/2

11. If the function
$f(x)=\begin{cases} -x, & x<1 \\ a+\cos^{-1}(x+b), & 1\le x\le 2 \end{cases}$ is differentiable at $x=1$, then $\frac{a}{b}$ is equal to

(a) $\frac{\pi-2}{2}$ (b) $\frac{-\pi-2}{2}$
(c) $\frac{\pi+2}{2}$ (d) $-1-\cos^{-1}(2)$

12. If the tangent at a point P, with parameter t, on the curves $x=4t^2+3$, $y=8t^3-1$, $t\in R$, meets the curve again at a point Q, then the coordinates of Q are

(a) $(t^2+3, -t^3-1)$ (b) $(4t^2+3, -8t^3-1)$
(c) (t^2+3, t^3-1) (d) $(16t^2+3, -64t^3-1)$

13. The minimum distance of a point on the curve $y=x^2-4$ from the origin is

(a) $\frac{\sqrt{19}}{2}$ (b) $\sqrt{\frac{15}{2}}$ (c) $\frac{\sqrt{15}}{2}$ (d) $\sqrt{\frac{19}{2}}$

14. If $\int \frac{dx}{\cos^3 x\sqrt{2\sin 2x}} = (\tan x)^A + C(\tan x)^B + K$, where K is a constant of integration, then $A+B+C$ is equal to

(a) $\frac{21}{5}$ (b) $\frac{16}{5}$ (c) $\frac{7}{10}$ (d) $\frac{27}{10}$

15. If $2\int_0^1 \tan^{-1}x\,dx = \int_0^1 \cot^{-1}(1-x+x^2)\,dx$, then $\int_0^1 \tan^{-1}(1-x+x^2)\,dx$ is equal to

(a) log 4 (b) $\frac{\pi}{2}+\log 2$ (c) log 2 (d) $\frac{\pi}{2}-\log 4$

16. The area (in sq units) of the region described by $A=\{(x,y)\,|\,y\ge x^2-5x+4,\ x+y\ge 1,\ y\le 0\}$ is

(a) $\frac{7}{2}$ (b) $\frac{19}{6}$ (c) $\frac{13}{6}$ (d) $\frac{17}{6}$

17. If $f(x)$ is a differentiable function in the interval $(0,\infty)$ such that $f(1)=1$ and $\lim_{t\to x}\frac{t^2f(x)-x^2f(t)}{t-x}=1$, for each $x>0$, then $f\left(\frac{3}{2}\right)$ is equal to

(a) $\frac{13}{6}$ (b) $\frac{23}{18}$ (c) $\frac{25}{9}$ (d) $\frac{31}{18}$

18. If a variable line drawn through the intersection of the lines $\frac{x}{3}+\frac{y}{4}=1$ and $\frac{x}{4}+\frac{y}{3}=1$, meets the coordinate axes at A and B, $(A\neq B)$, then the locus of the mid-point of AB is

(a) $6xy=7(x+y)$
(b) $4(x+y)^2-28(x+y)+49=0$
(c) $7xy=6(x+y)$
(d) $14(x+y)^2-97(x+y)+168=0$

19. The point (2, 1) is translated parallel to the line $L: x-y=4$ by $2\sqrt{3}$ units. If the new point Q lies in the third quadrant, then the equation of the line passing through Q and perpendicular to L is

(a) $x+y=2-\sqrt{6}$ (b) $x+y=3-3\sqrt{6}$
(c) $x+y=3-2\sqrt{6}$ (d) $2x+2y=1-\sqrt{6}$

20. A circle passes through $(-2,4)$ and touches the Y-axis at (0, 2). Which one of the following equations can represent a diameter of this circle?

(a) $4x+5y-6=0$ (b) $2x-3y+10=0$
(c) $3x+4y-3=0$ (d) $5x+2y+4=0$

21. Let a and b respectively be the semi-transverse and semi-conjugate axes of a hyperbola whose eccentricity satisfies the equation $9e^2-18e+5=0$. If $S(5,0)$ is a focus and $5x=9$ is the corresponding directrix of this hyperbola, then a^2-b^2 is equal to

(a) 7 (b) −7 (c) 5 (d) −5

22. If the tangent at a point on the ellipse $\frac{x^2}{27}+\frac{y^2}{3}=1$ meets the coordinate axes at A and B, and O is the origin, then the minimum area (in sq units) of the ΔOAB is

(a) $\frac{9}{2}$ (b) $3\sqrt{3}$ (c) $9\sqrt{3}$ (d) 9

23. The shortest distance between the lines $\frac{x}{2}=\frac{y}{2}=\frac{z}{1}$ and $\frac{x+2}{-1}=\frac{y-4}{8}=\frac{z-5}{4}$ lies in the interval

(a) [0, 1) (b) [1, 2) (c) (2, 3] (d) (3, 4]

24. The distance of the point $(1,-2,4)$ from the plane passing through the point (1, 2, 2) and perpendicular to the planes $x-y+2z=3$ and $2x-2y+z+12=0$, is

(a) $2\sqrt{2}$ (b) 2 (c) $\sqrt{2}$ (d) $\frac{1}{\sqrt{2}}$

25. In a ΔABC, right angled at the vertex A, if the position vectors of A, B and C are respectively $3\hat{i}+\hat{j}-\hat{k}$, $-\hat{i}+3\hat{j}+p\hat{k}$ and $5\hat{i}+q\hat{j}-4\hat{k}$, then the point (p, q) lies on a line

(a) parallel to X-axis
(b) parallel to Y-axis
(c) making an acute angle with the positive direction of X-axis
(d) making an obtuse angle with the positive direction of X-axis

26. If the mean deviation of the numbers $1, 1+d, \ldots, 1+100d$ from their mean is 255, then a value of d is

(a) 10.1 *(b)* 20.2 *(c)* 10 *(d)* 5.05

27. If A and B are any two events such that $P(A)=\frac{2}{5}$ and $P(A\cap B)=\frac{3}{20}$, then the conditional probability, $P(A/(A'\cup B'))$, where A' denotes the complement of A, is equal to

(a) $\frac{1}{4}$ *(b)* $\frac{5}{17}$ *(c)* $\frac{8}{17}$ *(d)* $\frac{11}{20}$

28. The number of $x\in[0, 2\pi]$ for which $\left|\sqrt{2\sin^4 x+18\cos^2 x}-\sqrt{2\cos^4 x+18\sin^2 x}\right|=1$ is

(a) 2 *(b)* 4 *(c)* 6 *(d)* 8

29. If m and M are the minimum and the maximum values of $4+\frac{1}{2}\sin^2 2x-2\cos^4 x$, $x\in R$, then $M-m$ is equal to

(a) $\frac{15}{4}$ *(b)* $\frac{9}{4}$
(c) $\frac{7}{4}$ *(d)* $\frac{1}{4}$

30. Consider the following two statements

P: If 7 is an odd number, then 7 is divisible by 2.

Q: If 7 is a prime number, then 7 is an odd number.

If V_1 is the truth value of the contrapositive of P and V_2 is the truth value of contrapositive of Q, then the ordered pair (V_1, V_2) is equal to

(a) (T, T) *(b)* (T, F) *(c)* (F, T) *(d)* (F, F)

Answers with Explanation

1. ***(b)*** We have, $f_0(x)=\frac{1}{1-x}$

and $f_{n+1}(x)=f_0(f_n(x)), \forall\ x\in R-\{0, 1\}$, where $n=0, 1, 2, \ldots$

Consider $f_1(x)=f_0(f_0(x))$

$$=\frac{1}{1-f_0(x)}=\frac{1}{1-\frac{1}{1-x}}=\frac{1-x}{(1-x)-1}$$

$$\Rightarrow \quad f_1(x)=\frac{x-1}{x} \qquad \ldots(i)$$

Now, consider $f_2(x)=f_0(f_1(x))$

$$=\frac{1}{1-f_1(x)}$$

$$=\frac{1}{1-\frac{(x-1)}{x}}=x$$

Clearly, $f_3(x)=f_0(f_2(x))=f_0(x)$ $[\because f_2(x)=x]$

$f_4(x)=f_0(f_3(x))=f_0(f_0(x))=f_1(x)$

$\vdots \qquad \vdots$

$f_{100}(x)=f_1(x)$

Hence, $f_{100}(3)+f_1\left(\frac{2}{3}\right)+f_2\left(\frac{3}{2}\right)$

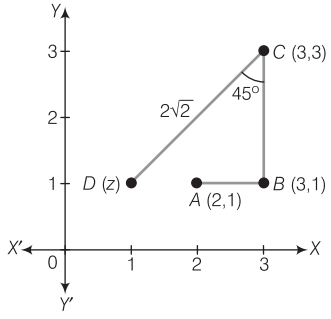
$$=f_1(3)+f_1\left(\frac{2}{3}\right)+f_2\left(\frac{3}{2}\right)$$

$$=\frac{2}{3}+\frac{\left(-\frac{1}{3}\right)}{\left(\frac{2}{3}\right)}+\frac{3}{2}=\frac{2}{3}-\frac{1}{2}+\frac{3}{2}$$

$$=\frac{4-3+9}{6}$$

$$=\frac{10}{6}=\frac{5}{3}$$

2. ***(b)*** Let the given point be $A(2+i)$. Then, according to given information, required point is D.



Now, let point D be represented by z in the argand plane.

Then, by rotation theorem, we have

$$\frac{z-(3+3i)}{(3+i)-(3+3i)}=\frac{|z-(3+3i)|}{|(3+i)-(3+3i)|}e^{-\frac{i\pi}{4}} \text{ (clockwise)}$$

$$\Rightarrow \quad \frac{z-(3+3i)}{-2i}=\frac{2\sqrt{2}}{2}\left(\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \quad \frac{z-(3+3i)}{-2i}=(1-i)$$

$$\Rightarrow \quad z-(3+3i)=-2i(1-i)=-2i-2$$

$$\Rightarrow \quad z=3+3i-2i-2 \Rightarrow z=1+i$$

Hence, the point D is represented by $(1+i)$.

3. ***(d)*** The given equations are

$$x^2+bx-1=0 \quad \text{and} \quad x^2+x+b=0$$

Let α be the common root. Then, we have

$$\alpha^2+b\alpha-1=0 \quad \text{and} \quad \alpha^2+\alpha+b=0$$

$$\Rightarrow \quad \frac{\alpha^2}{b^2+1}=\frac{-\alpha}{b+1}=\frac{1}{1-b}$$

$\Rightarrow \quad \alpha^2 = \dfrac{b^2+1}{1-b}$ and $\alpha = -\dfrac{(1+b)}{1-b} \Rightarrow \dfrac{(1+b)^2}{(1-b)^2} = \dfrac{b^2+1}{1-b}$

$\Rightarrow \quad (b+1)^2 = (b^2+1)(1-b)$

$\Rightarrow \quad b^2 + 1 + 2b = b^2 - b^3 + 1 - b$

$\Rightarrow \quad b^3 + 3b = 0 \Rightarrow b = 0$ or $b = \pm\sqrt{3}\, i$

Note that when $b = 0$, the common root is (-1). Hence, $b = \pm i\sqrt{3} \Rightarrow |b| = \sqrt{3}$

4. ***(d)*** We have, $P = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $Q = PAP^T$

Consider, $P^TQ^{2015}P = P^T(PAP^T)^{2015}P \qquad [\because Q = PAP^T]$

$= P^T(PAP^T)(PAP^T)(PAP^T)\ldots(PAP^T)P$

$= (P^TP)\cdot A(P^TP)\cdot A(P^TP)\cdot A \ldots A(P^TP)$

$= A^{2015} \qquad \left[\because P^TP = \frac{1}{4}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}\begin{bmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = I\right]$

Now, consider $A^2 = A\cdot A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$A^3 = A^2\cdot A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

$\vdots \qquad \vdots$

$\therefore \quad A^{2015} = \begin{bmatrix} 1 & 2015 \\ 0 & 1 \end{bmatrix}$

Hence, $P^TQ^{2015}P = \begin{bmatrix} 1 & 2015 \\ 0 & 1 \end{bmatrix}$

5. ***(c)*** Let $\Delta = \begin{vmatrix} \cos x & \sin x & \sin x \\ \sin x & \cos x & \sin x \\ \sin x & \sin x & \cos x \end{vmatrix}$

On applying $R_1 \to R_1 + R_2 + R_3$, we get

$$\Delta = \begin{vmatrix} \cos x + 2\sin x & 2\sin x + \cos x & 2\sin x + \cos x \\ \sin x & \cos x & \sin x \\ \sin x & \sin x & \cos x \end{vmatrix}$$

$$\Rightarrow \quad \Delta = (\cos x + 2\sin x)\begin{vmatrix} 1 & 1 & 1 \\ \sin x & \cos x & \sin x \\ \sin x & \sin x & \cos x \end{vmatrix}$$

On applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get

$$\Delta = (\cos x + 2\sin x)\begin{vmatrix} 1 & 0 & 0 \\ \sin x & \cos x - \sin x & 0 \\ \sin x & 0 & \cos x - \sin x \end{vmatrix}$$

$\Rightarrow \quad \Delta = (\cos x + 2\sin x)(\cos x - \sin x)^2$

Now, $\Delta = 0 \Rightarrow (\cos x + 2\sin x)(\cos x - \sin x)^2 = 0$

$\Rightarrow \quad \cos x = -2\sin x$ or $\cos x = \sin x$

$\Rightarrow \quad \tan x = -\dfrac{1}{2}$ or $\tan x = 1 \Rightarrow x = -\tan^{-1}\left(\dfrac{1}{2}\right)$ or $x = \dfrac{\pi}{4}$

Hence, two distinct real roots exist in the interval $\left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$.

6. ***(a)*** In the word 'MEDITERRANEAN', we have 1M, 3E, 1D, 1I, 1T, 2R, 2A, 2N, i.e. 13 letters.

Now, let us fix R as first letter and E as fourth letter in the 13 letters word.

R			E									

Clearly, required number of words

= Number of ways of arranging rest 11 letters

$= \dfrac{(11)!}{(2!)^3}$

7. ***(a)*** We have,

$(1+x)^{2016} + x(1+x)^{2015} + x^2(1+x)^{2014} + \ldots + x^{2016} = \sum_{i=0}^{2016} a_i x^i$

Clearly, LHS is a GP with first term, $a = (1+x)^{2016}$ and common ratio, $r = \dfrac{x}{1+x}$.

We have, $(1+x)^{2016}\left[\dfrac{1-\left(\frac{x}{1+x}\right)^{2017}}{1-\left(\frac{x}{1+x}\right)}\right] = \sum_{i=0}^{2016} a_i x^i$

$\Rightarrow \quad (1+x)^{2016}\left[\dfrac{(1+x)^{2017} - x^{2017}}{(1+x)^{2016}}\right] = \sum_{i=0}^{2016} a_i x^i$

$\Rightarrow \quad (1+x)^{2017} - x^{2017} = \sum_{i=0}^{2016} a_i x^i$

$\Rightarrow \quad ({}^{2017}C_0 + {}^{2017}C_1 x + {}^{2017}C_2 x^2 + \ldots + {}^{2017}C_{2017}x^{2017}) - x^{2017} = \sum_{i=0}^{2016} a_i x^i$

Now, a_{17} = Coefficient of $x^{17} = {}^{2017}C_{17} = \dfrac{(2017)!}{(17)!\,(2000)!}$

8. ***(d)*** We have, $x + y + z = 12$ and $x^3y^4z^5 = (0.1)(600)^3$

Consider, $\dfrac{3\left(\frac{x}{3}\right) + 4\left(\frac{y}{4}\right) + 5\left(\frac{z}{5}\right)}{12} \geq \left\{\left(\frac{x}{3}\right)^3 \cdot \left(\frac{y}{4}\right)^4 \cdot \left(\frac{z}{5}\right)^5\right\}^{\frac{1}{12}}$ $[\because \text{AM} \geq \text{GM}]$

$\Rightarrow \quad 1 \geq \left(\dfrac{x^3y^4z^5}{3^3\cdot 4^4\cdot 5^5}\right)^{1/12}$

$\Rightarrow \quad \dfrac{x^3y^4z^5}{3^3\cdot 4^3\cdot 5^3(4\cdot 5^2)} \leq 1 \Rightarrow \dfrac{x^3y^4z^5}{(60)^3(100)} \leq 1$

$\Rightarrow \quad \dfrac{x^3y^4z^5}{(600)^3(0.1)} \leq 1 \Rightarrow x^3y^4z^5 \leq (0.1)(600)^3$

But it is given that, $x^3y^4z^5 = (0.1)(600)^3$

$\therefore$ Equality exists between AM and GM.

$\Rightarrow$ The numbers $\dfrac{x}{3}, \dfrac{y}{4}, \dfrac{z}{5}$ are equal, i.e. we have

$\dfrac{x}{3} = \dfrac{y}{4} = \dfrac{z}{5} = k$ (say) $\Rightarrow x = 3k$, $y = 4k$ and $z = 5k$

$\because \quad x + y + z = 12$

$\therefore \quad k = 1 \Rightarrow x = 3$, $y = 4$ and $z = 5$

Hence, $x^3 + y^3 + z^3 = 3^3 + 4^3 + 5^3$

$= 27 + 64 + 125 = 216$

9. ***(b)*** Consider,

$$\sum_{r=1}^{15} r^2\left(\frac{{}^{15}C_r}{{}^{15}C_{r-1}}\right) = \sum_{r=1}^{15} r^2\left(\frac{15-r+1}{r}\right)\left[\because \frac{{}^{n}C_r}{{}^{n}C_{r-1}} = \frac{n-r+1}{r}\right]$$

$$= \sum_{r=1}^{15} r\,(16-r) = \sum_{r=1}^{15}(16r - r^2)$$

$$= 16\sum_{r=1}^{15} r - \sum_{r=1}^{15} r^2$$

$$= 16\cdot\frac{15(15+1)}{2} - \frac{15(15+1)(30+1)}{6}$$

$$= \frac{15\times 16}{2}\left[16 - \frac{31}{3}\right] = 120\times\frac{17}{3} = 40\times 17 = 680$$

10. ***(b)*** We have, $\lim\limits_{x\to\infty}\left(1+\frac{a}{x}-\frac{4}{x^2}\right)^{2x} = e^3$

$$\Rightarrow \quad e^{\lim\limits_{x\to\infty}\left(\frac{a}{x}-\frac{4}{x^2}\right)\cdot 2x} = e^3$$

If $\lim\limits_{x\to\infty} f(x) = 1$ and $\lim\limits_{x\to a} g(x) = \infty$, then

$$\lim_{x\to a}(f(x))^{g(x)} = e^{\lim\limits_{x\to a}(f(x)-1)\cdot g(x)}$$

$$\Rightarrow \quad e^{\lim\limits_{x\to\infty}\left(2a-\frac{8}{x}\right)} = e^3$$

$$\Rightarrow \quad e^{2a} = e^3$$

$$\Rightarrow \quad 2a = 3$$

$$\Rightarrow \quad a = \frac{3}{2}$$

11. ***(c)*** We have, $f(x) = \begin{cases} -x, & x<1 \\ a+\cos^{-1}(x+b), & 1\le x\le 2 \end{cases}$

is differentiable at $x = 1$.

We have, $\quad Lf'(1) = Rf'(1) = f'(1) \qquad \ldots(i)$

Clearly, $\quad f'(x) = \begin{cases} -1, & x<1 \\ \dfrac{-1}{\sqrt{1-(x+b)^2}}, & 1<x<2 \end{cases}$

$$\Rightarrow \quad Lf'(1) = -1 \text{ and } Rf'(1) = \frac{-1}{\sqrt{1-(1+b)^2}}$$

$$\because \quad Lf'(1) = Rf'(1)$$

$$\therefore \quad -1 = \frac{-1}{\sqrt{1-(1+b)^2}}$$

$$\Rightarrow \quad 1-(1+b)^2 = 1$$

$$\Rightarrow \quad (1+b)^2 = 0 \;\Rightarrow\; b = -1$$

Also, as f is differentiable at $x = 1$, therefore f is continuous at $x = 1$.

$$\Rightarrow \quad \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$

$$\Rightarrow \quad -1 = a + \cos^{-1}(1+b)$$

$$\Rightarrow \quad -1 = a + \cos^{-1}(0) \qquad [\because b = -1]$$

$$\Rightarrow \quad -1 = a + \frac{\pi}{2}$$

$$\Rightarrow \quad a = -\left(\frac{\pi}{2}+1\right)$$

Hence, $\quad \dfrac{a}{b} = \dfrac{-\left(\frac{\pi+2}{2}\right)}{(-1)} = \dfrac{\pi+2}{2}$

12. ***(a)*** Given equations of curve are

$$x = 4t^2 + 3,\ y = 8t^3 - 1$$

Clearly, coordinates of P are $(4t^2+3,\ 8t^3-1)$.

Now, let us first find the equation of tangent at P. For this consider

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{24t^2}{8t} = 3t$$

Now, equation of tangent at P is given by

$$(y-(8t^3-1)) = \frac{dy}{dx}(x-(4t^2+3))$$

$$\Rightarrow \quad (y-(8t^3-1)) = 3t(x-(4t^2+3)) \qquad \ldots(i)$$

Now, let the coordinates of Q be $(4t_1^2+3,\ 8t_1^3-1)$.

Then, these coordinates will satisfy Eq.(i).

So, we get $\quad ((8t_1^3-1)-(8t^3-1)) = 3t((4t_1^2+3)-(4t^2+3))$

$$\Rightarrow \quad 8(t_1^3 - t^3) = 3t(4(t_1^2 - t^2))$$

$$\Rightarrow \quad 2(t_1-t)(t_1^2+t^2+t_1t) = 3t(t_1-t)(t_1+t)$$

$$\Rightarrow \quad 2t_1^2 + 2t^2 + 2t_1t = 3t_1t + 3t^2$$

$$\Rightarrow \quad 2t_1^2 = t_1t + t^2$$

$$\Rightarrow \quad 2t_1^2 - t_1t - t^2 = 0$$

$$\Rightarrow \quad (t_1-t)(2t_1+t) = 0$$

$$\Rightarrow \quad t_1 = -\frac{t}{2} \qquad [\because t_1 \neq t]$$

Hence, coordinates of Q are $(t^2+3,\ -t^3-1)$.

13. ***(c)*** Let $P(x, y)$ be any point on the curve $y = x^2 - 4$. Then,

$$OP = \sqrt{x^2+y^2}$$

$$\Rightarrow \quad OP^2 = x^2 + y^2$$

$$\Rightarrow \quad OP^2 = y^2 + y + 4$$

Now, let $D(y) = OP^2 = y^2 + y + 4$

For minimum, put $D'(y) = 0$

$$\Rightarrow \quad 2y + 1 = 0 \Rightarrow y = -\frac{1}{2}$$

$$\Rightarrow \quad x = \pm\sqrt{\frac{7}{2}} \qquad [\because y = x^2 - 4]$$

$\therefore$ Points on the curves are $\left(\pm\sqrt{\frac{7}{2}}, -\frac{1}{2}\right)$, which are at a minimum distance from origin.

Hence, required distance $= \sqrt{\frac{7}{2}+\frac{1}{4}} = \sqrt{\frac{15}{4}} = \frac{\sqrt{15}}{2}$

14. ***(b)*** We have,

$$\int \frac{dx}{\cos^3 x\sqrt{2\sin 2x}} = (\tan x)^A + C(\tan x)^B + K \qquad \ldots(i)$$

Now, $\quad$ let $I = \int \dfrac{dx}{\cos^3 x\sqrt{2\sin 2x}} = \dfrac{1}{2}\int \dfrac{dx}{\cos^3 x\sqrt{\sin x\cos x}}$

On dividing numerator and denominator by $\cos^4 x$, we get

$$I = \frac{1}{2}\int \frac{\sec^4 x\,dx}{\sqrt{\tan x}} = \frac{1}{2}\int \frac{\sec^2 x(1+\tan^2 x)\,dx}{\sqrt{\tan x}}$$

Now, put $\tan x = t$

$$\Rightarrow \quad \sec^2 x\,dx = dt$$

$$\therefore \quad I = \frac{1}{2}\int \frac{(1+t^2)}{\sqrt{t}}dt = \frac{1}{2}\int [t^{-1/2} + t^{3/2}]\,dt$$

$$= \frac{1}{2}\left[2\sqrt{t} + \frac{2}{5}t^{5/2}\right] + K = (\tan x)^{1/2} + \frac{1}{5}(\tan x)^{5/2} + K$$

On comparing it with Eq. (i), we get

$$A = \frac{1}{2}$$

$$B = \frac{5}{2}$$

and $$C = \frac{1}{5}$$

Hence, $A + B + C = \frac{1}{2} + \frac{5}{2} + \frac{1}{5} = \frac{16}{5}$

15. ***(c)*** We have,

$$2\int_0^1 \tan^{-1} x\, dx = \int_0^1 \cot^{-1}(1 - x + x^2)\, dx \quad \ldots(i)$$

Consider,

$$\int_0^1 \tan^{-1}(1 - x + x^2)\, dx = \int_0^1 \left(\frac{\pi}{2} - \cot^{-1}(1 - x + x^2)\right) dx$$

$$= \int_0^1 \frac{\pi}{2} dx - \int_0^1 \cot^{-1}(1 - x + x^2)\, dx$$

$$= \frac{\pi}{2} - 2\int_0^1 \tan^{-1} x\, dx \qquad \text{[using Eq. (i)]}$$

$$= \frac{\pi}{2} - 2\left[\tan^{-1} x \cdot x - \int \frac{1}{1 + x^2} \cdot x\, dx\right]_0^1$$

$$= \frac{\pi}{2} - 2\,[x \cdot \tan^{-1} x]_0^1 + \int_0^1 \frac{2x}{1 + x^2} dx$$

$$= \frac{\pi}{2} - 2\left[\frac{\pi}{4}\right] + [\log(1 + x^2)]_0^1$$

$$= \log 2$$

16. ***(b)*** Given equation of curve is

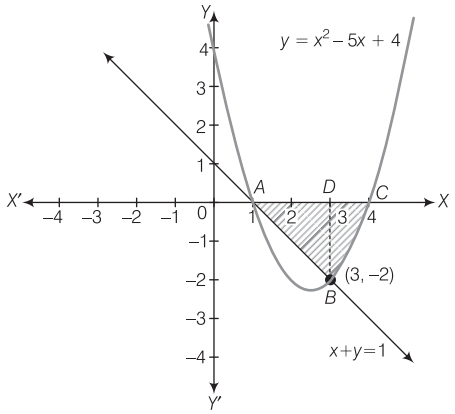
$$y = x^2 - 5x + 4$$

$$\Rightarrow \quad y = \left(x - \frac{5}{2}\right)^2 - \frac{9}{4}$$

$$\Rightarrow \quad \left(y + \frac{9}{4}\right) = \left(x - \frac{5}{2}\right)^2$$

which is parabola with vertex $\left(\frac{5}{2}, -\frac{9}{4}\right)$ and axis parallel to Y-axis.

Now, required region is the shaded region, shown in the figure given below.



Hence, required area = Area of region $ABDA$ + Area of region $BCDB$

$$= \left|\int_1^3 (1 - x)\, dx\right| + \left|\int_3^4 (x^2 - 5x + 4)\, dx\right|$$

$$= \left|\left[x - \frac{x^2}{2}\right]_1^3\right| + \left|\left[\frac{x^3}{3} - \frac{5x^2}{2} + 4x\right]_3^4\right|$$

$$= \left|\left(3 - \frac{9}{2}\right) - \frac{1}{2}\right| + \left|\left(\frac{64}{3} - 40 + 16 - 9 + \frac{45}{2} - 12\right)\right|$$

$$= |3 - 5| + \left|\frac{64}{3} + \frac{45}{2} - 45\right|$$

$$= 2 + \left|\frac{64}{3} - \frac{45}{2}\right| = 2 + \left|\frac{128 - 135}{6}\right|$$

$$= 2 + \frac{7}{6} = \frac{19}{6} \text{ sq units}$$

17. ***(d)*** Given, $\lim_{t \to x} \frac{t^2 f(x) - x^2 f(t)}{t - x} = 1 \qquad \left[\frac{0}{0} \text{ form}\right]$

$$\Rightarrow \lim_{t \to x} 2t\, f(x) - x^2 f'(t) = 1 \quad \text{[using L' Hospital's rule]}$$

$$\Rightarrow \quad 2x f(x) - x^2 f'(x) = 1$$

$$\Rightarrow \quad f'(x) = \frac{2x f(x) - 1}{x^2} \Rightarrow \frac{dy}{dx} = 2\frac{y}{x} - \frac{1}{x^2}$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{2}{x} \cdot y = -\frac{1}{x^2}$$

which is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$,

where $P = -\frac{2}{x}$ and $Q = -\frac{1}{x^2}$.

Now, IF $= e^{\int P dx} = e^{\int -\frac{2}{x} dx} = e^{-2\log x} = \frac{1}{x^2}$

and solution of differential equation is given by

$$y \cdot \left(\frac{1}{x^2}\right) = \int -\frac{1}{x^2} \cdot \frac{1}{x^2} dx + C \Rightarrow \frac{y}{x^2} = -\int x^{-4}\, dx + C$$

$$\Rightarrow \quad \frac{y}{x^2} = \frac{1}{3} x^{-3} + C$$

$$\because \quad f(1) = 1$$

$$\therefore \quad 1 = \frac{1}{3} + C \Rightarrow C = \frac{2}{3}$$

Thus, $$f(x) = \frac{1}{3x} + \frac{2}{3}x^2$$

$$\Rightarrow \quad f\left(\frac{3}{2}\right) = \frac{2}{9} + \frac{2}{3} \times \frac{9}{4} = \frac{2}{9} + \frac{3}{2} = \frac{4 + 27}{18} = \frac{31}{18}$$

18. ***(c)*** Given equation of lines are

$$4x + 3y = 12 \quad \ldots(i)$$

and $$3x + 4y = 12 \quad \ldots(ii)$$

Now, the equation of line passing through the intersection of above lines is given by

$$(4x + 3y) + \lambda(3x + 4y) = 12 + 12\lambda$$

$$\Rightarrow \quad (4 + 3\lambda)x + (3 + 4\lambda)y = 12(1 + \lambda)$$

$$\Rightarrow \quad \frac{4 + 3\lambda}{12(1 + \lambda)} x + \frac{3 + 4\lambda}{12(1 + \lambda)} y = 1$$

Clearly, this will intersect X-axis at $\left[\frac{12(1 + \lambda)}{4 + 3\lambda}, 0\right]$ and Y-axis at $\left[0, \frac{12(1 + \lambda)}{3 + 4\lambda}\right]$.

Thus, the coordinates of A and B are $\left[\frac{12(1+\lambda)}{4+3\lambda}, 0\right]$ and $\left[0, \frac{12(1+\lambda)}{3+4\lambda}\right]$, respectively.

Now, let (h, k) be the mid-point of AB. Then,

$$h = \frac{6(1+\lambda)}{4+3\lambda} \text{ and } k = \frac{6(1+\lambda)}{3+4\lambda}$$

$$\Rightarrow \quad \lambda = \frac{6-4h}{3h-6} \qquad ...(iii)$$

and $$\lambda = \frac{6-3k}{4k-6} \qquad ...(iv)$$

From Eqs. (iii) and (iv), we get

$$\frac{6-4h}{3h-6} = \frac{6-3k}{4k-6}$$

$$\Rightarrow \quad (6-4h)(4k-6) = (6-3k)(3h-6)$$

$$\Rightarrow 24k - 36 - 16hk + 24h = 18h - 36 - 9hk + 18k$$

$$\Rightarrow \quad 6k + 6h - 7hk = 0 \Rightarrow 7hk = 6(h+k)$$

Hence, the locus of mid-point (h,k) is $7xy = 6(x+y)$.

19. ***(c)*** Given equation of line is

$$x - y = 4 \qquad ...(i)$$

Let $\tan\theta$ be the slope of line parallel to given line. Then,

$$\tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$$

Now, the coordinates of any point on the line, passing through (2, 1) and having slope 1, may be taken as $(2 \pm r\cos\theta, 1 \pm r\sin\theta)$ i.e. $\left(2 \pm r\cdot\frac{1}{\sqrt{2}}, 1 \pm r\cdot\frac{1}{\sqrt{2}}\right)$, where r is the distance of point on the line from the point (2, 1).

Here, $r = 2\sqrt{3}$

$\therefore$ Coordinates of Q are $(2-\sqrt{6}, 1-\sqrt{6})$.

[$\because Q$ lies in third quadrant]

Hence, required equation of line is given by

$$(y-(1-\sqrt{6})) = -1(x-(2-\sqrt{6}))$$

$$\Rightarrow \quad y - 1 + \sqrt{6} = -x + 2 - \sqrt{6}$$

$$\Rightarrow \quad x + y = 3 - 2\sqrt{6}$$

20. ***(b)*** Clearly, the centre of circle will lie on the line $y = 2$.

$\therefore$ Coordinates of centre will be of the form $(h, 2)$.

Now, let the equation of circle be $(x-h)^2 + (y-2)^2 = r^2$

$\because$ It is passes through (0, 2) and (– 2, 4).

So, we have,

$$h^2 = r^2 \qquad ...(i)$$

and $$(2+h)^2 + 4 = r^2 \qquad ...(ii)$$

Now, from Eqs. (i) and (ii), we get

$$(2+h)^2 + 4 = h^2$$

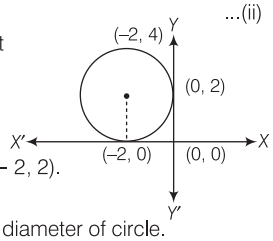
$$\Rightarrow \quad 4 + h^2 + 4h + 4 = h^2$$

$$\Rightarrow \quad 4h = -8 \Rightarrow h = -2$$

Thus, coordinates of centre are (– 2, 2).

$\because$ It satisfy $2x - 3y + 10 = 0$.

$\therefore 2x - 3y + 10 = 0$ represents a diameter of circle.



21. ***(b)*** Given, $9e^2 - 18e + 5 = 0$

$$\Rightarrow \quad 9e^2 - 15e - 3e + 5 = 0$$

$$\Rightarrow \quad 3e(3e-5) - 1(3e-5) = 0$$

$$\Rightarrow \quad (3e-5)(3e-1) = 0$$

$$\Rightarrow \quad e = \frac{5}{3} \qquad [\because e > 1]$$

Also, we have $ae = 5$

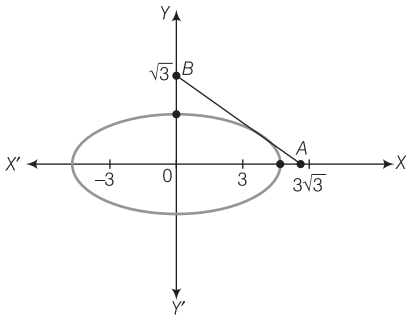
$$\Rightarrow \quad a\cdot\frac{5}{3} = 5 \Rightarrow a = 3$$

Now, consider $b^2 = a^2(e^2 - 1)$

$$= 9\left(\frac{25}{9} - 1\right) = 16$$

Hence, $a^2 - b^2 = 9 - 16 = -7$

22. ***(d)*** Given equation of ellipse is $\frac{x^2}{27} + \frac{y^2}{3} = 1$.



$\therefore$ Any point on the ellipse will be of the form $(3\sqrt{3}\cos\theta, \sqrt{3}\sin\theta)$.

Now, equation of tangent to the ellipse at $(3\sqrt{3}\cos\theta, \sqrt{3}\sin\theta)$ is

$$\frac{x\cdot 3\sqrt{3}\cos\theta}{27} + \frac{y\cdot\sqrt{3}\sin\theta}{3} = 1$$

$$\Rightarrow \quad \frac{x}{3\sqrt{3}\sec\theta} + \frac{y}{\sqrt{3}\,\text{cosec}\,\theta} = 1$$

Thus, the coordinates A and B are $(3\sqrt{3}\sec\theta, 0)$ and $(0, \sqrt{3}\,\text{cosec}\,\theta)$.

Hence, area of $\Delta OAB = \frac{1}{2}\cdot 3\sqrt{3}\sec\theta\cdot\sqrt{3}\,\text{cosec}\,\theta = \frac{9}{\sin 2\theta}$

$\Rightarrow$ Minimum area = 9 [$\because$ maximum value of $\sin 2\theta$ is 1]

23. ***(c)*** Given equation of lines are

$$\frac{x}{2} = \frac{y}{2} = \frac{z}{1} \qquad ...(i)$$

and $$\frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4} \qquad ...(ii)$$

On comparing these equations with

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$$

and $$\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$$, we get

$x_1 = y_1 = z_1 = 0$; $x_2 = -2$, $y_2 = 4$, $z_2 = 5$

$a_1 = 2$, $b_1 = 2$, $c_1 = 1$; $a_2 = -1$, $b_2 = 8$, $c_2 = 4$

Now, shortest distance

$$= \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}$$

$$= \frac{\begin{vmatrix} -2 & 4 & 5 \\ 2 & 2 & 1 \\ -1 & 8 & 4 \end{vmatrix}}{\sqrt{(8-8)^2 + (-1-8)^2 + (16+2)^2}}$$

$$= \frac{-2(0) - 4(9) + 5(18)}{\sqrt{9^2 + 18^2}} = \frac{54}{\sqrt{405}} = \frac{54}{9\sqrt{5}} = \frac{6}{\sqrt{5}},$$

which lies in (2, 3].

24. ***(a)*** Clearly, equation of plane, passing through (1, 2, 2) and perpendicular to the planes $x - y + 2z = 3$ and $2x - 2y + z + 12 = 0$, is given by

$$\begin{vmatrix} x-1 & y-2 & z-2 \\ 1 & -1 & 2 \\ 2 & -2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad (x-1)(-1+4) - (y-2)(1-4) + (z-2)(0) = 0$$

$$\Rightarrow \quad 3(x-1) + 3(y-2) = 0$$

$$\Rightarrow \quad x - 1 + y - 2 = 0$$

$$\Rightarrow \quad x + y - 3 = 0$$

Now, required distance

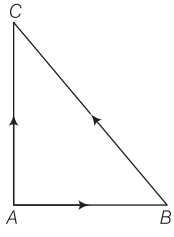
$$= \frac{|1 + (-2) - 3|}{\sqrt{1^2 + 1^2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

25. ***(c)*** We have,

$$\mathbf{OA} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$$

$$\mathbf{OB} = -\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + p\hat{\mathbf{k}}$$

and $\mathbf{OC} = 5\hat{\mathbf{i}} + q\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$



Clearly, $\mathbf{AB} = \mathbf{OB} - \mathbf{OA}$

$$= -4\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + (p+1)\hat{\mathbf{k}}$$

$$\mathbf{BC} = \mathbf{OC} - \mathbf{OB}$$

$$= 6\hat{\mathbf{i}} + (q-3)\hat{\mathbf{j}} - (p+4)\hat{\mathbf{k}}$$

and $\mathbf{AC} = \mathbf{OC} - \mathbf{OA} = 2\hat{\mathbf{i}} + (q-1)\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$

$\because$ It is given that $\angle A = 90°$

$$\therefore \quad \mathbf{AB} \cdot \mathbf{AC} = 0$$

$$\Rightarrow \quad [-4\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + (p+1)\hat{\mathbf{k}}] \cdot [2\hat{\mathbf{i}} + (q-1)\hat{\mathbf{j}} - 3\hat{\mathbf{k}}] = 0$$

$$\Rightarrow \quad -8 + 2(q-1) - 3(p+1) = 0$$

$$\Rightarrow \quad -8 + 2q - 2 - 3p - 3 = 0$$

$$\Rightarrow \quad 2q - 3p = 13$$

Thus, (p, q) lies on the line $3x - 2y + 13 = 0$, which makes an acute angle with positive direction of X-axis.

26. ***(a)*** Given numbers are $1, 1 + d, \ldots, 1 + 100d$

and their MD $(\bar{x}) = 255$

Clearly, mean $(\bar{x})$ of given numbers $= \dfrac{\frac{101}{2}[1 + (1 + 100d)]}{101}$

$$= \frac{1}{2}[2 + 100d] = 1 + 50d$$

and the sum of absolute deviation of numbers from mean is

$50d + 49d + 48d + \ldots + 0 + d \ldots + 50d$

$= 2(d + 2d + \ldots + 50d) = 2d(1 + 2 + \ldots + 50)$

$$= 2d \cdot \frac{50 \times 51}{2} = 50 \times 51 \times d$$

Hence, MD $(\bar{x}) = \dfrac{50 \times 51 \times d}{101} = 255$ [given]

$$\Rightarrow \quad d = \frac{255 \times 101}{50 \times 51} = \frac{505}{50} = 10.1$$

27. ***(b)*** Given, $P(A) = \dfrac{2}{5}$ and $P(A \cap B) = \dfrac{3}{20}$

To find $P(A / (A' \cup B'))$

Consider, $P(A / (A' \cup B')) = \dfrac{P(A \cap (A' \cup B'))}{P(A' \cup B')}$

$$= \frac{P[(A \cap A') \cup (A \cap B')]}{P((A \cap B)')} = \frac{P[\phi \cup (A \cap B')]}{1 - P(A \cap B)}$$

$$= \frac{P(A \cap B')}{1 - P(A \cap B)} = \frac{P(A) - P(A \cap B)}{1 - P(A \cap B)} = \frac{\frac{2}{5} - \frac{3}{20}}{1 - \frac{3}{20}} = \frac{\frac{5}{20}}{\frac{17}{20}} = \frac{5}{17}$$

28. ***(d)*** We have,

$$\left|\sqrt{2\sin^4 x + 18\cos^2 x} - \sqrt{2\cos^4 x + 18\sin^2 x}\right| = 1$$

$$\Rightarrow \sqrt{2\sin^4 x + 18\cos^2 x} - \sqrt{2\cos^4 x + 18\sin^2 x} = \pm 1$$

Taking +ve sign,

$$\sqrt{2\sin^4 x + 18\cos^2 x} - \sqrt{2\cos^4 x + 18\sin^2 x} = 1$$

$$\Rightarrow \sqrt{2\sin^4 x + 18\cos^2 x} = 1 + \sqrt{2\cos^4 x + 18\sin^2 x}$$

On squaring both sides, we get

$$2\sin^4 x + 18\cos^2 x = 1 + 2\cos^4 x + 18\sin^2 x + 2\sqrt{2\cos^4 x + 18\sin^2 x}$$

$$\Rightarrow \quad 2(\sin^4 x - \cos^4 x) + 18(\cos^2 x - \sin^2 x) = 1 + 2\sqrt{2\cos^4 x + 18\sin^2 x}$$

$$\Rightarrow \quad 2(\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x) - 18(\sin^2 x - \cos^2 x) = 1 + 2\sqrt{2\cos^4 x + 18\sin^2 x}$$

$$\Rightarrow \quad 2(\sin^2 x - \cos^2 x) - 18(\sin^2 x - \cos^2 x) = 1 + 2\sqrt{2\cos^4 x + 18\sin^2 x}$$

$$[\because \sin^2 x + \cos^2 x = 1]$$

$$\Rightarrow \quad -16(\sin^2 x - \cos^2 x) = 1 + 2\sqrt{2\cos^4 x + 18\sin^2 x}$$

$$\Rightarrow -16(1 - \cos^2 x - \cos^2 x) = 1 + 2\sqrt{2\cos^4 x + 18(1 - \cos^2 x)}$$

$$[\because \sin^2 x = 1 - \cos^2 x]$$

$$\Rightarrow \quad -16(1 - 2\cos^2 x) = 1 + 2\sqrt{2\cos^4 x + 18 - 18\cos^2 x}$$

$$\Rightarrow \quad -16 + 32\cos^2 x = 1 + 2\sqrt{2\cos^4 x - 18\cos^2 x + 18}$$

$$\Rightarrow \quad -17 + 32\cos^2 x = 2\sqrt{2\cos^4 x - 18\cos^2 x + 18}$$

Again, squaring both sides, we get

$$4(2\cos^4 x - 18\cos^2 x + 18) = 289 + 1024\cos^4 x - 1088\cos^2 x$$

$$\Rightarrow \quad 8\cos^4 x - 72\cos^2 x + 72 = 289 + 1024\cos^4 x - 1088\cos^2 x$$

$$\Rightarrow \quad 8\cos^4 x - 1024\cos^4 x - 72\cos^2 x + 1088\cos^2 x + 72 - 289 = 0$$

$$\Rightarrow \quad -1016\cos^4 x + 1016\cos^2 x - 217 = 0$$

$$\Rightarrow \quad 1016\cos^4 x + 1016\cos^2 x - 217 = 0$$

Since, degree of this equation is 4.

So, number of roots of $\cos x$ is 4.

Since, given equation is in modulus form, therefore total number of $x \in [0, 2\pi]$ is 8.

29. ***(b)*** Consider, $4 + \frac{1}{2}\sin^2 2x - 2\cos^4 x$

$$= 4 + \frac{1}{2}(2\sin x \cos x)^2 - 2\cos^4 x$$

$$= 4 + 2\sin^2 x \cos^2 x - 2\cos^4 x$$

$$= 4 + 2(1 - \cos^2 x)\cdot\cos^2 x - 2\cos^4 x$$

$$= 4 + 2\cos^2 x - 2\cos^4 x - 2\cos^4 x$$

$$= 4 + 2\cos^2 x - 4\cos^4 x$$

$$= -4\left[\cos^4 x - 2\cdot\frac{1}{4}\cos^2 x - 1\right]$$

$$= -4\left[(\cos^2 x)^2 - 2\cdot\frac{1}{4}\cos^2 x + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 - 1\right]$$

$$= -4\left[\left(\cos^2 x - \frac{1}{4}\right)^2 - \frac{17}{16}\right]$$

$$= \frac{17}{4} - 4\left(\cos^2 x - \frac{1}{4}\right)^2 = \frac{17}{4} - \left(2\cos^2 x - \frac{1}{2}\right)^2$$

$$= \frac{17}{4} - \left(1 + \cos 2x - \frac{1}{2}\right)^2 = \frac{17}{4} - \left(\cos 2x + \frac{1}{2}\right)^2$$

Now, M = Maximum value $= \frac{17}{4} - 0 = \frac{17}{4}$

[$\because$ maximum value occur, when $\left(\cos 2x + \frac{1}{2}\right)^2$ is minimum]

and m = minimum value $= \frac{17}{4} - \frac{9}{4} = 2$

[$\because$ minimum value occur, when $\left(\cos 2x + \frac{1}{2}\right)^2$ is maximum]

Hence, $M - m = \frac{17}{4} - 2 = \frac{9}{4}$

30. ***(c)*** We have the following statements

P: If 7 is an odd number, then 7 is divisible by 2

Q : If 7 is a prime number, then 7 is an odd number.

Clearly, the contrapositive of P is

"If 7 is not divisible by 2, then 7 is not an odd number."

and contrapositive of Q is

"If 7 is not an odd number, then 7 is not a prime number."

$\therefore$ $V_1 = F$ and $V_2 = T$

Hence, $(V_1, V_2) = (F, T)$

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UPSEE 2016
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- Use BLUE or BLACK BALL POINT PEN only for all entries and for filling the bubbles in the OMR Answer Sheet.
- Before opening the SECURITY SEAL of the question booklet, write your Name, Roll Number (In figures), OMR Answer Sheet Number in the space provided at the top of the Question Booklet. Non-compliance of these instructions would mean that the Answer Sheet cannot be evaluated leading the disqualification of the candidate.
- Each question carries FOUR marks. No marks will be awarded for unattempted questions. There is no negative marking on wrong answer.
- Each multiple choice questions has only one correct answer and marks shall be awarded for correct answer.
- Use of calculator, log table, mobile phones, any electronic gadget and slide rule etc., is strictly prohibited.
- Candidate will be allowed to leave the examination hall at the end of examination time period only.
- If a candidate is found in possession of books or any other printed or written material from which he/she might derive assistance, he/she is liable to be treated at disqualified. Similarly, if a candidate is found giving or obtaining (or attempting to give or obtain) assistance from any source, he/she is liable to be disqualified.
- English version of questions paper is to be considered as authentic and final to resolve any ambiguity.
- One blank sheet for rough work is also enclosed.
- OMR sheet is placed within this paper and can be taken out from this paper but seal of paper must be opened only at the start of paper.

EXAM CRUX

- This year Mathematics question paper was quite balanced as 30% questions were HOTS, 40% questions were moderate and 30% questions were easy. Overall, it was difficult as compared to previous year paper.
- According to branchwise division, it is found that 7 questions are asked from *Coordinate Geometry*; 13 questions are asked from *Algebra*; 13 questions are asked from *Calculus*; 4 questions are asked from *Probability*; 5 questions are asked from *Trigonometry*; 4 questions are asked from *Dynamics*; 2 questions are asked from *Statics* and 2 questions are asked from *Vector*.

1. The value of $\lim_{x \to 0} \frac{\cosh x - \cos x}{x \sin x}$ is

(a) 1/3 (b) 2 (c) 1 (d) 1/2

2. The value of maxima of $\left(\frac{1}{x}\right)^x$ is

(a) $\left(\frac{1}{e}\right)^e$ (b) e^e (c) e (d) $e^{1/e}$

3. The value of $\int_0^{1/\sqrt{2}} \frac{\sin^{-1} x}{(1-x^2)^{3/2}} dx$ is

(a) $\frac{\pi}{2} - \log 2$ (b) $\frac{\pi}{4} - \frac{1}{2}\log 2$

(c) $\frac{\pi}{2} + \frac{1}{2}\log 2$ (d) $\pi - \frac{1}{2}\log 2$

4. Integral of $\frac{1}{2 + \cos x}$ is

(a) $\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{1}{2}\tan x\right) + C$ (b) $\frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right) + C$

(c) $-\sin x \log(2 + \cos x) + C$ (d) $\sin x \log(2 + \cos x) + C$

5. The eccentricity of an ellipse $9x^2 + 16y^2 = 144$ is

(a) $\frac{\sqrt{3}}{5}$ (b) $\frac{\sqrt{5}}{3}$ (c) $\frac{\sqrt{7}}{4}$ (d) $\frac{2}{5}$

6. Taking axes of hyperbola as coordinate axes, find its equation when the distance between the foci is 16 and eccentricity is $\sqrt{2}$.

(a) $x^2 - y^2 = 32$ (b) $x^2 - y^2 = 64$

(c) $x^2 - y^2 = 8$ (d) $x^2 - y^2 = 16$

7. For a circle $x^2 + y^2 = 81$, what is the equation of chord whose mid-point is $(-2, 3)$?

(a) $2x - 3y + 13 = 0$ (b) $3x - 2y + 13 = 0$
(c) $2x - 3y - 13 = 0$ (d) $2x + 3y + 13 = 0$

8. The condition so that the line $lx + my + n = 0$ may touch the parabola $y^2 = 8x$, is

(a) $8m^2 = ln$ (b) $2m^2 = ln$ (c) $m^2 = 8ln$ (d) $m^2 = 2ln$

9. The equation of that diameter of the circle $x^2 + y^2 - 6x + 2y - 8 = 0$ which passes through the origin, is

(a) $x + 3y = 0$ (b) $3x - y = 0$
(c) $6x - y = 0$ (d) $3x + 2y = 0$

10. If z is a complex number, then $(z + 5)(\bar{z} + 5)$ is equal to

(a) $|z + 5i|^2$ (b) $|z - 5|^2$ (c) $(z + 5)^2$ (d) $|z + 5|^2$

11. If z is a complex number, then which of the following statement is true?

(a) $(z\bar{z})$ is purely imaginary (b) $(z\bar{z})$ is non-negative real
(c) $(z - \bar{z})$ is purely real (d) $(z + \bar{z})$ is purely imaginary

12. If ω is the cubic root of unity, then the value of $(1 + \omega - \omega^2)^2 + (1 - \omega + \omega^2)^2 + 1$ is

(a) -1 (b) 7 (c) 1 (d) -3

13. If $(1 + i\sqrt{3})^{12} = a + ib$, where a and b are real, then the value of b is

(a) $(\sqrt{3})^{12}$ (b) $(2)^{12}$ (c) 0 (d) 1

14. If $f(\theta) = 2(\sec^2\theta + \cos^2\theta)$, then its value always

(a) $4 > f(\theta) > 2$ (b) $f(\theta) \geq 4$
(c) $f(\theta) < 2$ (d) $f(\theta) = 2$

15. If $\cot x - \tan x = 2$, then generalised solution is (here, n is integer)

(a) $x = \frac{n\pi}{2} + \frac{\pi}{8}$ (b) $x = \frac{n\pi}{4} + \frac{\pi}{16}$
(c) $x = 2n\pi + \frac{\pi}{2}$ (d) $x = n\pi + \frac{\pi}{4}$

16. A plane is flying horizontally at a height of 1 km from ground. Angle of elevation of the plane at a certain instant is 60°. After 20 s, angle of elevation is found 30°. The speed of plane is

(a) $100\sqrt{3}$ m/s (b) $200\sqrt{3}$ m/s
(c) $\frac{100}{\sqrt{3}}$ m/s (d) $\frac{200}{\sqrt{3}}$ m/s

17. $\sin^2\theta\cos^3\theta - \sin^4\theta\cos\theta$ is equal to

(a) $\frac{1}{2}\sin^2 2\theta$ (b) $\frac{1}{4}\sin\theta\sin 4\theta$
(c) $\frac{1}{2}\cos\theta\sin 4\theta$ (d) $\frac{1}{4}\cos\theta\sin 4\theta$

18. If $2\sin C\cos A = \sin B$, then ΔABC is

(a) right angled triangle (b) isosceles triangle
(c) equilateral triangle (d) None of these

19. The value of $\tan\left[\frac{1}{2}\cos^{-1}\left(\frac{2}{3}\right)\right]$ is

(a) $\frac{1}{\sqrt{5}}$ (b) $\sqrt{\frac{3}{10}}$ (c) $\sqrt{\frac{5}{2}}$ (d) $1 - \sqrt{\frac{5}{2}}$

20. If $r^2 = x^2 + y^2 + z^2$ and $\tan^{-1}\frac{yz}{xr} + \tan^{-1}\frac{xz}{yr} = \frac{\pi}{2} - \tan^{-1}\phi$, then

(a) $\phi = \frac{zr}{xy}$ (b) $\phi = \frac{xy}{zr}$
(c) $\phi = \frac{x + y}{zr}$ (d) $\phi = \frac{yz}{xr} + \frac{xz}{yr}$

21. Consider digits 1, 2, 3, 4, 5, 6 and 7. Using these digits, numbers of five digits are formed. Then, probability of these such five digit number that have odd digits at their both ends, is

(a) $\frac{3}{7}$ (b) $\frac{1}{7}$
(c) $\frac{2}{7}$ (d) None of these

22. Out of 100 bicycles, 10 bicycles have puncture. What is the probability of not having any punctured bicycle in a sample of 5 bicycles?

(a) $\frac{1}{2^9}$ (b) $\left(\frac{9}{10}\right)^5$ (c) $\frac{1}{10^5}$ (d) $\frac{1}{2^5}$

23. Probability of solving a particular question by person A is $\frac{1}{3}$ and probability of solving that question by person B is $\frac{2}{5}$. What is the probability of solving that question by atleast one of them?

(a) $\frac{3}{5}$ (b) $\frac{7}{9}$ (c) $\frac{2}{5}$ (d) $\frac{2}{3}$

24. Four men and three women are standing in a line for railway ticket. The probability of standing them in alternate manner is

(a) $\frac{1}{84}$ (b) $\frac{1}{7}$ (c) $\frac{1}{35}$ (d) $\frac{1}{33}$

25. $\log_3 2$, $\log_6 2$, $\log_{12} 2$ are in

(a) HP (b) AP
(c) GP (d) None of these

26. If p, q, r, s, t and u are in AP, then difference $(t - r)$ is equal to

(a) $2(s - r)$ (b) $(u - q)$ (c) $2(s - p)$ (d) $2(u - q)$

27. The value of $[(\log_b a)(\log_c b)(\log_a c)]$ is

(a) abc (b) $\log abc$ (c) 0 (d) 1

28. If $p = \frac{1}{\log_3 \pi} + \frac{1}{\log_4 \pi} + 1$, then

(a) $2.5 < p < 3$ (b) $p > 3$
(c) $1.5 < p < 2$ (d) $2 < p < 2.5$

29. In the expansion of $\left(\frac{3x^2}{5}+\frac{5}{3x^2}\right)^{10}$, mid term is

(a) 291 (b) 242 (c) 252 (d) 284

30. If roots of equation $x^2 + x + 1 = 0$ are a, b and roots of $x^2 + px + q = 0$ are $\frac{a}{b}, \frac{b}{a}$; then the value of $p + q$ is

(a) 2 (b) $\frac{\sqrt{2}+1}{2}$ (c) – 1 (d) 1

31. The value of determinant $\begin{vmatrix} 1/a & bc & a^3 \\ 1/b & ca & b^3 \\ 1/c & ab & c^3 \end{vmatrix}$ is

(a) $a^2b^2c^2(a-b)(b-c)(c-a)$
(b) 0
(c) $(a-b)(b-c)(c-a)$
(d) None of the above

32. If $\begin{bmatrix} 3 & -1 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} 3x \\ 1 \end{bmatrix}+\begin{bmatrix} -2x \\ 3 \end{bmatrix}=\begin{bmatrix} 8 \\ 9 \end{bmatrix}$, then the value of x is

(a) $-\frac{3}{8}$ (b) 7
(c) $-\frac{2}{9}$ (d) None of these

33. Consider A and B two square matrices of same order. Select the correct alternative.

(a) $|AB|$ must be greater than $|A|$
(b) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is not unit matrix
(c) $|A + B|$ must be greater than $|A|$
(d) If $AB = 0$, either A or B must be zero matrix

34. Function $f : N \to N$, $f(x) = 2x + 3$ is

(a) many-one onto function
(b) many-one into function
(c) one-one onto function
(d) one-one into function

35. If domain of the function $f(x) = x^2 - 6x + 7$ is $(-\infty, \infty)$, then its range is

(a) $[-2, 3]$ (b) $(-\infty, -2)$
(c) $(-\infty, \infty)$ (d) $[-2, \infty)$

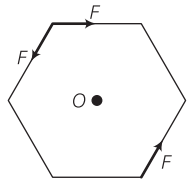
36. The resultant of two forces P and Q is of magnitude P. If the force P is doubled, Q remaining the same, then angle between new resultant and the force Q is

(a) 60° (b) 90°
(c) 30° (d) 45°

37. The centre of gravity (centre of mass) of a rod (of length L), whose linear mass density varies as the square of the distance from one end is at

(a) $\frac{3L}{5}$ (b) $\frac{2L}{5}$
(c) $\frac{L}{3}$ (d) $\frac{3L}{4}$

38. Three forces each of magnitude F are applied along the edges of a regular hexagon as shown in the figure. Each side of hexagon is a. What is the resultant moment (torque) of these three forces about centre O?



(a) $\frac{3\sqrt{3}}{2}aF$ (b) $\frac{1}{2}aF$ (c) $3aF$ (d) $\frac{\sqrt{3}}{2}aF$

39. The coordinates of a moving point particle in a plane at time t is given by $x = a(t + \sin t)$, $y = a(1 - \cos t)$. The magnitude of acceleration of the particle is

(a) $2a$ (b) $\frac{\sqrt{3}}{2}a$ (c) a (d) $\sqrt{3}a$

40. A point particle moves along a straight line such that $x = \sqrt{t}$, where t is time. Then, ratio of acceleration to cube of the velocity is

(a) – 1 (b) – 0.5 (c) – 3 (d) – 2

41. A body of mass m falls from rest through a height h under gravitation acceleration g and is then brought to rest by penetrating through a depth d into some sand. The average deceleration of the body during penetration into sand is

(a) $\frac{gh^2}{d^2}$ (b) $\frac{gh^2}{2d^2}$ (c) $\frac{gh}{d}$ (d) $\frac{gd}{h}$

42. A normal is drawn at a point (x_1, y_1) of the parabola $y^2 = 16x$ and this normal makes equal angle with both X and Y-axes. Then, point (x_1, y_1) is

(a) (4, – 8) (b) (1, – 4) (c) (4, – 4) (d) (2, – 8)

43. Two vectors $\mathbf{A} = 3$ and $\mathbf{B} = 4$ are perpendicular. Resultant of both these vectors is $\mathbf{R}$. The projection of the vector $\mathbf{B}$ on the vector $\mathbf{R}$ is

(a) 5 (b) 1.25 (c) 3.2 (d) 2.4

44. A vector $\mathbf{R}$ is given by $\mathbf{R} = \mathbf{A} \times (\mathbf{B} \times \mathbf{C})$, which of the following is true?

(a) $\mathbf{R}$ must be perpendicular to $\mathbf{B}$
(b) $\mathbf{R}$ is parallel to $\mathbf{A}$
(c) $\mathbf{R}$ must be parallel to $\mathbf{B}$
(d) None of the above

45. The solution of the differential equation $\frac{dy}{dx} = 2e^{x-y} + x^2e^{-y}$ is

(a) $e^y = 2e^x + \frac{x^3}{3} + C$ (b) $e^{-y} = 2e^x + \frac{x^{-3}}{3} + C$
(c) $e^{-y} = 2e^x + \frac{x^3}{3} + C$ (d) $e^y = 2e^{-x} + \frac{x^3}{3} + C$

46. The solution of the differential equation $(x + 2y^3)\frac{dy}{dx} = y$ is

(a) $y^3 + Cx = y$ (b) $\frac{xy^4}{2} + xy = Cy$

(c) $y^3 + Cy = x$ (d) $x + 2y^3 = y + C$

47. The value of the following expression $\lim_{n\to\infty} \frac{1}{n^3}(1^2 + 2^2 + 3^2 + \ldots + n^2)$ is

(a) $\frac{1}{2}$ (b) $\frac{2}{3}$ (c) $\frac{1}{3}$ (d) $\frac{1}{6}$

48. If function $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right); & x \neq 0 \\ a \quad ; & x = 0 \end{cases}$ is continuous at $x = 0$, then the value of a is

(a) 0 (b) 1

(c) – 1 (d) None of these

49. The derivative of $y = x^{\sin x}$ is

(a) $x^{\sin x}\left(\cos x \log x + \frac{\sin x}{x}\right)$ (b) $\cos x \log x + \frac{\sin x}{x}$

(c) $\cos x\, x^{\sin x - 1}$ (d) $\frac{\sin 2x}{2} x^{\sin x - 1}$

50. The tangents to curve $y = x^3 - 2x^2 + x - 2$ which are parallel to straight line $y = x$, are

(a) $x + y = 2$ and $x - y = \frac{86}{27}$

(b) $x - y = 2$ and $x - y = \frac{86}{27}$

(c) $x - y = 2$ and $x + y = \frac{86}{27}$

(d) $x + y = 2$ and $x + y = \frac{86}{27}$

Answers with Explanation

1. ***(c)*** Let $I = \lim_{x\to 0} \frac{\cosh x - \cos x}{x\sin x}$ $\left[\frac{0}{0}\text{ form}\right]$

Now, applying L'Hospital's rule, we get

$$I = \lim_{x\to 0} \frac{\sinh x + \sin x}{x\cos x + \sin x} \quad \left[\because \frac{d}{dx}(\cosh x) = \sinh x\right]$$

Again, applying L'Hospital's rule, we get

$$I = \lim_{x\to 0} \frac{\cosh x + \cos x}{x(-\sin x) + \cos x + \cos x}$$

$$\left[\because \frac{d}{dx}(\sinh x) = \cosh x\right]$$

$$\Rightarrow \quad I = \lim_{x\to 0} \frac{\cos h x + \cos x}{2\cos x - x\sin x} = \frac{\cosh(0) + \cos(0)}{2\cos(0) - 0}$$

$$= \frac{1+1}{2} = 1 \qquad [\because \cosh(0) = 1 \text{ and } \cos(0) = 1]$$

$$\Rightarrow \quad I = 1$$

Alternate Solution

Consider, $\lim_{x\to 0} \frac{\cosh x - \cos x}{x\sin x} = \lim_{x\to 0}\left[\frac{\frac{e^x + e^{-x}}{2} - \cos x}{x\sin x}\right]$

$$= \lim_{x\to 0}\left[\frac{\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\right) - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right)}{x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)}\right]$$

$$\left[\because e^x + e^{-x} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots \right.$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$\left.\text{and} \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right]$$

$$= \lim_{x\to 0}\left[\frac{2\left(\frac{x^2}{2!} + \frac{x^6}{6!} + \ldots\right)}{x^2\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots\right)}\right]$$

$$= \lim_{x\to 0}\left[\frac{2\left(\frac{1}{2!} + \frac{x^4}{6!} + \ldots\right)}{\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots\right)}\right]$$

$$= \frac{2\left(\frac{1}{2!} + 0 + 0 + \ldots\right)}{(1 - 0 + 0 + \ldots)}$$

$$= 2 \times \frac{1}{2} = 1$$

2. ***(d)*** Let $y = \left(\frac{1}{x}\right)^x$

On taking log both sides, we get

$$\log y = x \log \frac{1}{x} = -x\log x$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = -\left[x\cdot\frac{1}{x} + \log x\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = -\left(\frac{1}{x}\right)^x (1 + \log x) = 0 \qquad \ldots(i)$$

Now, $1 + \log x = 0$

$\Rightarrow \quad \log x = -1$

$\Rightarrow \quad x = e^{-1}$

Again, differentiating Eq. (i) both sides, we get

$$\frac{d^2y}{dx^2} = -\left[\left(\frac{1}{x}\right)^x \frac{d}{dx}(1 + \log x) + (1 + \log x)\frac{d}{dx}\left(\frac{1}{x}\right)^x\right]$$

$$= -\left[\left(\frac{1}{x}\right)^x \cdot \frac{1}{x} + (1+\log x)\cdot(1+\log x)\left(-\left(\frac{1}{x}\right)^x\right)\right]$$

[from Eq. (i)]

$$= -\left(\frac{1}{x}\right)^x \cdot \frac{1}{x} + \left(\frac{1}{x}\right)^x (1+\log x)\cdot(1+\log x)$$

$$\therefore \frac{d^2y}{dx^2} = \left(\frac{1}{x}\right)^x (1+\log x)\cdot(1+\log x) - \left(\frac{1}{x}\right)^x \left[\frac{1}{x}\right]$$

Now, $\left[\frac{d^2y}{dx^2}\right]_{x=e^{-1}} = -e^{e^{-1}}\cdot e < 0$

Thus, the function is maximum when $x = e^{-1}$.

Now, maximum value of function

$$= \left(\frac{1}{e^{-1}}\right)^{e^{-1}} = e^{1/e}$$

3. ***(b)*** Let $I = \int_0^{1/\sqrt{2}} \frac{\sin^{-1} x}{(1-x^2)^{3/2}}\,dx$

Put $\sin^{-1} x = t \Rightarrow x = \sin t$

$\Rightarrow \quad dx = \cos t\,dt$

When $x = 0$, then $\sin t = 0 = \sin 0 \Rightarrow t = 0$

When $x = \frac{1}{\sqrt{2}}$, then $\sin t = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}$

$\Rightarrow \quad t = \frac{\pi}{4}$

$$\therefore \quad I = \int_0^{\pi/4} \frac{t}{\cos^3 t}\cdot\cos t\,dt$$

$$= \int_0^{\pi/4} t\sec^2 t\,dt$$

$$= [t\tan t]_0^{\pi/4} - \int_0^{\pi/4}\tan t\,dt$$

$$= \frac{\pi}{4} - [\log\sec t]_0^{\pi/4} = \frac{\pi}{4} - \log\sqrt{2}$$

$$= \frac{\pi}{4} - \log 2^{1/2} = \frac{\pi}{4} - \frac{1}{2}\log 2$$

4. ***(b)*** Let $I = \int \frac{1}{2+\cos x}\,dx$

$$\Rightarrow \quad I = \int \frac{1}{2+\left(\frac{1-\tan^2 x/2}{1+\tan^2 x/2}\right)}\,dx \qquad \left[\because \cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}\right]$$

$$\Rightarrow \quad I = \int \frac{\sec^2 x/2}{2+2\tan^2 x/2 + 1 - \tan^2 x/2}\,dx$$

$$\Rightarrow \quad I = \int \frac{\sec^2 x/2}{3+\tan^2 x/2}\,dx$$

Put $\quad \tan\frac{x}{2} = t$

$\Rightarrow \quad \frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$

$$\therefore \quad I = 2\int \frac{dt}{3+t^2}$$

$$= 2\int \frac{dt}{(\sqrt{3})^2 + (t)^2}$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}} + C = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{\sqrt{3}}\right) + C$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right) + C$$

5. ***(c)*** Given equation of an ellipse is

$$9x^2 + 16y^2 = 144$$

$$\Rightarrow \quad \frac{9x^2}{144} + \frac{16y^2}{144} = \frac{144}{144}$$

$$\Rightarrow \quad \frac{x^2}{16} + \frac{y^2}{9} = 1$$

Here, $\quad a^2 = 16, b^2 = 9 \qquad [\because a^2 > b^2]$

$$\therefore \text{ Eccentricity } (e) = \sqrt{1-\frac{b^2}{a^2}} = \sqrt{1-\frac{9}{16}}$$

$$= \sqrt{\frac{7}{16}} = \frac{\sqrt{7}}{4}$$

6. ***(a)*** Given, $2ae = 16$ and $e = \sqrt{2}$

$$\Rightarrow \quad a = \frac{16}{2\sqrt{2}} \Rightarrow a = 4\sqrt{2}$$

Also, $\quad e = \sqrt{1+\frac{b^2}{a^2}}$

$$\Rightarrow \quad e^2 = 1 + \frac{b^2}{a^2}$$

$$\Rightarrow \quad 2 = 1 + \frac{b^2}{32} \Rightarrow 1 = \frac{b^2}{32}$$

$$\Rightarrow \quad b^2 = 32 \qquad \Rightarrow b = 4\sqrt{2}$$

$\therefore$ Required equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{32} - \frac{y^2}{32} = 1$$

$$\Rightarrow \quad x^2 - y^2 = 32$$

7. ***(a)*** $\because$ Equation of the required chord is $T = S'$

$$\therefore \quad -2x + 3y - 9 = 4 + 9 - 9$$

$$\Rightarrow \quad -2x + 3y = 4 + 9$$

$$\Rightarrow \quad 2x - 3y + 13 = 0$$

8. ***(b)*** Given equation of the line is $lx + my + n = 0$

$$\Rightarrow \quad my = -lx - n$$

$$\Rightarrow \quad y = \frac{-l}{m}x - \frac{n}{m}$$

On comparing with $y = Mx + C$, we get

$$M = \frac{-l}{m} \text{ and } C = \frac{-n}{m} \qquad \ldots(i)$$

Again, given equation of the parabola is $y^2 = 8x$

On comparing with $y^2 = 4ax$, we get

$$4a = 8 \Rightarrow a = 2$$

Now, by condition of tangency, $C = \frac{a}{M}$

$$\Rightarrow \quad \frac{-n}{m} = \frac{2}{-l/m} \qquad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad \frac{n}{m} = \frac{2m}{l} \Rightarrow ln = 2m^2$$

9. ***(a)*** Given equation of a circle is

$$x^2 + y^2 - 6x + 2y - 8 = 0$$

$\therefore$ Centre $= (3, -1)$

Now, the diameter also passes through the origin $O(0, 0)$.

So, the required equation of the diameter is

$$y - 0 = \frac{-1 - 0}{3 - 0}(x - 0)$$

$$\Rightarrow \quad y = -\frac{1}{3}(x - 0)$$

$$\Rightarrow \quad 3y = -x$$

$$\Rightarrow \quad x + 3y = 0$$

10. ***(d)*** $(z + 5)(\bar{z} + 5) = z \cdot \bar{z} + 5(z + \bar{z}) + 25$

$$= |z|^2 + 2 \cdot 5|z| + 5^2$$

$$= |z + 5|^2$$

11. ***(b)*** Let $z = x + iy$, then $\bar{z} = x - iy$

Now,

(a) $z\bar{z} = x^2 + y^2$ [purely real]

(b) $z\bar{z} = x^2 + y^2$ [non-negative real]

(c) $z - \bar{z} = 2i\,\text{Im}(z)$ [purely imaginary]

(d) $z + \bar{z} = 2\,\text{Re}(z)$ [purely real]

12. ***(d)*** $(1 + \omega - \omega^2)^2 + (1 - \omega + \omega^2)^2 + 1$

$$= (-2\omega^2)^2 + (-2\omega)^2 + 1$$

$$= 4(\omega^4 + \omega^2) + 1 \qquad [\because 1 + \omega + \omega^2 = 0, \omega^3 = 1]$$

$$= 4(\omega \cdot \omega^3 + \omega^2) + 1$$

$$= 4(\omega + \omega^2) + 1$$

$$= 4(-1) + 1 = -4 + 1 = -3$$

13. ***(c)*** We have, $(1 + i\sqrt{3})^{12} = a + ib$

Now, $\left[2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]^{12} = a + ib$ [De-Moivre's theorem]

$$\Rightarrow \quad 2^{12}\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right]^{12} = a + ib$$

$$\Rightarrow \quad 4096[\cos 4\pi + i\sin 4\pi] = a + ib$$

$$\Rightarrow \quad 4096[1 + 0] = a + ib$$

$$\Rightarrow \quad 4096 = a + ib$$

On comparing, we get $b = 0$

14. ***(b)*** Given, $f(\theta) = 2(\sec^2\theta + \cos^2\theta)$ ...(i)

Let $a = 2\sec^2\theta$ and $b = 2\cos^2\theta$

We know that, $\frac{a + b}{2} \geq \sqrt{ab}$

$$2\sec^2\theta + 2\cos^2\theta \geq 2\sqrt{2\sec^2\theta \cdot 2\cos^2\theta}$$

$$\Rightarrow \quad 2(\sec^2\theta + \cos^2\theta) \geq 2\sqrt{4}$$

$$\Rightarrow \quad 2(\sec^2\theta + \cos^2\theta) \geq 4$$

$$\therefore \quad f(\theta) \geq 4 \qquad \text{[from Eq. (i)]}$$

15. ***(a)*** We have, $\cot x - \tan x = 2$

$$\Rightarrow \quad \frac{\cos x}{\sin x} - \frac{\sin x}{\cos x} = 2$$

$$\Rightarrow \quad \frac{\cos^2 x - \sin^2 x}{\sin x \cos x} = 2$$

$$\Rightarrow \quad \cos 2x = 2\sin x \cos x$$

$$\Rightarrow \quad \cos 2x = \sin 2x$$

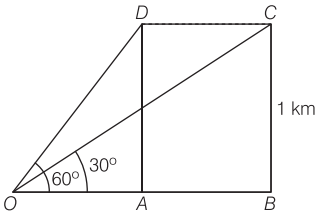
$$\Rightarrow \quad \tan 2x = 1 = \tan\frac{\pi}{4}$$

$$\Rightarrow \quad 2x = n\pi + \frac{\pi}{4}, n \in Z$$

$$\Rightarrow \quad x = \frac{n\pi}{2} + \frac{\pi}{8}, n \in Z$$

16. ***(c)*** Let AD be the height at which the plane is flying i.e. 1 km = 1000 m and C be the position of the plane after 20 s.

It is given that, $\angle AOD = 60^\circ$ and $\angle BOC = 30^\circ$



In right angled ΔOAD and ΔOBC, we have

$$\tan 60^\circ = \frac{AD}{OA} \text{ and } \tan 30^\circ = \frac{BC}{OB}$$

$$\Rightarrow \quad OA = \frac{1000}{\sqrt{3}} \text{ and } \frac{1}{\sqrt{3}} = \frac{1000}{OB} \qquad [\because AD = BC = 1000\text{ m}]$$

$$\Rightarrow \quad OA = \frac{1000}{\sqrt{3}}\text{m and } OB = 1000\sqrt{3}\text{m}$$

Now, $OA + AB = 1000\sqrt{3}$

$$\Rightarrow \quad AB = 1000\sqrt{3} - \frac{1000}{\sqrt{3}}$$

$$\Rightarrow \quad AB = \frac{3000 - 1000}{\sqrt{3}} = \frac{2000}{\sqrt{3}}\text{ m}$$

$$\therefore \text{ Speed} = \frac{\text{Distance}}{\text{Time}} = \frac{2000}{\sqrt{3} \times 20} = \frac{100}{\sqrt{3}}\text{ m/s}$$

17. ***(b)*** $\sin^2\theta\cos^3\theta - \sin^4\theta\cos\theta$

$$= \sin^2\theta\cos\theta(\cos^2\theta - \sin^2\theta)$$

$$= \sin^2\theta\cos\theta(\cos 2\theta)$$

$$= \sin\theta\sin\theta\cos\theta(\cos 2\theta)$$

$$= \sin\theta \cdot \frac{2}{2}\sin\theta\cos\theta(\cos 2\theta)$$

$$= \frac{1}{2}\sin\theta\sin 2\theta\cos 2\theta \qquad [\because 2\sin\theta\cos\theta = \sin 2\theta]$$

$$= \frac{1}{2}\sin\theta \cdot \frac{2}{2}\sin 2\theta\cos 2\theta = \frac{1}{4}\sin\theta\sin 4\theta$$

18. ***(b)*** In ΔABC, $2\sin C\cos A = \sin B$ [given]

$$\Rightarrow \quad \sin(C + A) + \sin(C - A) = \sin B$$

$$[\because 2\sin A\cos B = \sin(A + B) + \sin(A - B)]$$

$$\Rightarrow \quad \sin(180^\circ - B) + \sin(C - A) = \sin B$$

$$\Rightarrow \quad \sin B + \sin(C - A) = \sin B \quad [\because \angle A + \angle B + \angle C = 180^\circ]$$

$$\Rightarrow \quad \sin(C - A) = 0 = \sin 0^\circ$$

$$\Rightarrow \quad C - A = 0$$

$$\Rightarrow \quad C = A$$

Hence, ΔABC is an isosceles triangle.

19. ***(a)*** We have, $\tan\left[\frac{1}{2}\cos^{-1}\left(\frac{2}{3}\right)\right]$

Put $\cos^{-1}\frac{2}{3} = \theta \Rightarrow \cos\theta = \frac{2}{3}$

$\therefore \tan\left\{\frac{1}{2}\cos^{-1}\left(\frac{2}{3}\right)\right\} = \tan\frac{\theta}{2}$

$$= \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \sqrt{\frac{1-2/3}{1+2/3}} = \sqrt{\frac{1/3}{5/3}} = \frac{1}{\sqrt{5}}$$

20. ***(b)*** We have, $\tan^{-1}\frac{yz}{xr} + \tan^{-1}\frac{xz}{yr} = \frac{\pi}{2} - \tan^{-1}\phi$

Now, $\tan^{-1}\left(\frac{\frac{yz}{xr}+\frac{xz}{yr}}{1-\frac{yz}{xr}\cdot\frac{xz}{yr}}\right) = \frac{\pi}{2} - \tan^{-1}\phi$

$\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$

$\Rightarrow \tan^{-1}\left[\frac{y^2zr + x^2zr}{xyr^2 - xyz^2}\right] = \frac{\pi}{2} - \tan^{-1}\phi$

$\Rightarrow \tan^{-1}\left[\frac{zr(x^2+y^2)}{xy(r^2-z^2)}\right] = \frac{\pi}{2} - \tan^{-1}\phi$

$\Rightarrow \tan^{-1}\frac{zr(x^2+y^2)}{xy(x^2+y^2)} = \frac{\pi}{2} - \tan^{-1}\phi$

$[\because x^2 + y^2 = r^2 - z^2]$

$\Rightarrow \tan^{-1}\frac{zr}{xy} = \cot^{-1}\phi$ $\left[\because \tan^{-1}\theta + \cot^{-1}\theta = \frac{\pi}{2}\right]$

$\Rightarrow \tan^{-1}\frac{zr}{xy} = \tan^{-1}\frac{1}{\phi}$ $\left[\because \tan^{-1}\frac{1}{\theta} = \cot^{-1}\theta\right]$

$\Rightarrow \frac{1}{\phi} = \frac{zr}{xy} \Rightarrow \phi = \frac{xy}{zr}$

21. ***(c)*** $\because$ Total outcomes $= {}^7P_5$

and favourable outcomes $= {}^4P_2 \times {}^5P_3$

$\because$ Probability $= \frac{\text{Favourable outcomes}}{\text{Total outcomes}}$

$\therefore$ Required probability $= \frac{{}^4P_2 \times {}^5P_3}{{}^7P_5} = \frac{\frac{4!}{2!}\times\frac{5!}{2!}}{\frac{7!}{2!}} = \frac{2}{7}$

22. ***(b)*** Here, $n = 5, r = 0, p = \frac{10}{100}, q = \frac{90}{100}$

Now, $P(X = 0) = {}^nC_r p^r q^{n-r}$

$$= {}^5C_0\left(\frac{10}{100}\right)^0\left(\frac{90}{100}\right)^{5-0}$$

$$= 1\times 1\times\left(\frac{9}{10}\right)^5 = \left(\frac{9}{10}\right)^5$$

23. ***(a)*** Given, $P(A) = \frac{1}{3}$ and $P(B) = \frac{2}{5}$

$\therefore$ $P(\bar{A}) = 1 - P(A) = 1 - \frac{1}{3} = \frac{2}{3}$

and $P(\bar{B}) = 1 - P(B) = 1 - \frac{2}{5} = \frac{3}{5}$

Now, P(atleast one of them solve that question)

$= P(A \cup \bar{B}) + P(\bar{A} \cup B) + P(A \cup B)$

$= P(A)\cdot P(\bar{B}) + P(\bar{A})\cdot P(B) + P(A)\cdot P(B)$

[$\because A$ and B are independent events]

$$= \frac{1}{3}\cdot\frac{3}{5} + \frac{2}{3}\cdot\frac{2}{5} + \frac{1}{3}\cdot\frac{2}{5}$$

$$= \frac{3}{15} + \frac{4}{15} + \frac{2}{15} = \frac{9}{15} = \frac{3}{5}$$

24. ***(c)*** $\because$ Total outcomes $= {}^7P_7 = 7!$

and favourable outcomes $= 3! \times 4!$

$\therefore$ Required probability $= \frac{\text{Favourable outcomes}}{\text{Total outcomes}}$

$$= \frac{3!\times 4!}{7!} = \frac{3\times 2\times 1\times 4!}{7\times 6\times 5\times 4!} = \frac{1}{35}$$

25. ***(a)*** We have, $\log_3 2, \log_6 2, \log_{12} 2$

Let $a = \log_3 2 = \frac{\log 2}{\log 3}$ $\left[\because \log_m n = \frac{\log n}{\log m}\right]$

$b = \log_6 2 = \frac{\log 2}{\log 6}$ and $c = \log_{12} 2 = \frac{\log 2}{\log 12}$

Now, $\frac{1}{a} + \frac{1}{c} = \frac{\log 3}{\log 2} + \frac{\log 12}{\log 2}$

$$= \frac{\log 3 + \log 12}{\log 2} = \frac{\log 36}{\log 2}$$

$[\because \log(mn) = \log m + \log n]$

$$= \frac{\log 6^2}{\log 2} = \frac{2\log 6}{\log 2} = \frac{2}{b}$$

Hence, a, b and c are in HP.

26. ***(a)*** Given that, p, q, r, s, t and u are in AP.

$\therefore$ $s - r = t - s = u - t$

By taking first two terms,

$s - r = t - s$

$\Rightarrow$ $s - r - r = t - s - r$ [by subtracting r from both sides]

$\Rightarrow$ $s - 2r = (t - r) - s$

$\Rightarrow$ $2s - 2r = t - r$

$\therefore$ $2(s - r) = t - r$

27. ***(d)*** $(\log_b a)(\log_c b)(\log_a c)$

$$= \frac{\log a}{\log b}\times\frac{\log b}{\log c}\times\frac{\log c}{\log a}$$ $\left[\because \log_m n = \frac{\log n}{\log m}\right]$

$= 1$

28. ***(b)*** Given, $p = \frac{1}{\log_3\pi} + \frac{1}{\log_4\pi} + 1$

Now, $p = \log_\pi 3 + \log_\pi 4 + 1$

$$= \frac{\log 3}{\log\pi} + \frac{\log 4}{\log\pi} + 1$$

$$= \frac{\log 3 + \log 4 + \log\pi}{\log\pi} = \frac{\log(3\times 4\times\pi)}{\log\pi}$$

$$= \frac{\log 12\pi}{\log\pi} = \log_\pi 12\pi = 3.1707$$

$\therefore$ $p > 3$

29. ***(c)*** Given expansion $= \left(\frac{3x^2}{5} + \frac{5}{3x^2}\right)^{10}$

On comparing with $(x + a)^n$, we get

$$x = \frac{3x^2}{5},\ a = \frac{5}{3x^2} \text{ and } n = 10$$

$\because$ Number of terms $= 10 + 1 = 11$

$\therefore$ Mid term $= \left(\frac{11+1}{2}\right)$th term $=$ 6th term

$$T_6 = T_{5+1} = {}^{10}C_5 \left[\frac{3x^2}{5}\right]^{10-5} \left[\frac{5}{3x^2}\right]^5$$

$[\because T_{r+1} = {}^nC_r x^{n-r} a^r]$

$$= {}^{10}C_5 \left[\frac{3x^2}{5}\right]^5 \left[\frac{5}{3x^2}\right]^5 = {}^{10}C_5 = 252$$

30. ***(a)*** Given, roots of the equation $x^2 + x + 1 = 0$ are a and b.

$\therefore$ Sum of roots, $a + b = \frac{-\text{Coefficient of } x}{\text{Coefficient of } x^2} = -1$

and product of roots,

$$ab = \frac{\text{Constant term}}{\text{Coefficient of } x^2} = 1 \quad \ldots(\text{i})$$

Again, roots of the equation $x^2 + px + q = 0 = \frac{a}{b}$ and $\frac{b}{a}$.

$\therefore$ Sum of roots, $\frac{a}{b} + \frac{b}{a} = -p$

and product of roots, $\frac{a}{b} \cdot \frac{b}{a} = q$

$\Rightarrow$ $1 = q$ $\quad \ldots(\text{ii})$

Now, $\frac{a}{b} + \frac{b}{a} = -p$

$\Rightarrow$ $\frac{a^2 + b^2}{ab} = -p$

$\Rightarrow$ $\frac{(a+b)^2 - 2ab}{ab} = -p$

$\Rightarrow$ $\frac{(-1)^2 - 2(1)}{1} = -p$ [from Eq. (i)]

$\Rightarrow$ $\frac{1-2}{1} = -p \Rightarrow p = 1$

$\therefore$ $p + q = 1 + 1 = 2$

31. ***(b)*** Let $\Delta = \begin{vmatrix} 1/a & bc & a^3 \\ 1/b & ca & b^3 \\ 1/c & ab & c^3 \end{vmatrix}$

Applying $R_1 \to aR_1, R_2 \to bR_2, R_3 \to cR_3$, we get

$$\Delta = \frac{1}{abc} \begin{vmatrix} 1 & abc & a^4 \\ 1 & abc & b^4 \\ 1 & abc & c^4 \end{vmatrix}$$

Taking abc common from C_2, we get

$$\Delta = \frac{abc}{abc} \begin{vmatrix} 1 & 1 & a^4 \\ 1 & 1 & b^4 \\ 1 & 1 & c^4 \end{vmatrix}$$

$= 0$ $[\because C_1$ and C_2 are identical$]$

32. ***(d)*** Given, $\begin{bmatrix} 3 & -1 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} 3x \\ 1 \end{bmatrix} + \begin{bmatrix} -2x \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \end{bmatrix}$

$$\text{LHS} = \begin{bmatrix} 3 & -1 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} 3x \\ 1 \end{bmatrix} + \begin{bmatrix} -2x \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 9x + (-1) \\ 0 + 6 \end{bmatrix} + \begin{bmatrix} -2x \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 9x - 1 \\ 6 \end{bmatrix} + \begin{bmatrix} -2x \\ 3 \end{bmatrix} = \begin{bmatrix} 9x - 1 - 2x \\ 6 + 3 \end{bmatrix} = \begin{bmatrix} 7x - 1 \\ 9 \end{bmatrix}$$

Now, $\begin{bmatrix} 7x - 1 \\ 9 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \end{bmatrix}$

$\therefore$ $7x - 1 = 8$

$\Rightarrow$ $7x = 9 \Rightarrow x = \frac{9}{7}$

33. ***(b)***

34. ***(d)*** Let x_1 and x_2 be any two elements of N.

$\therefore$ $f(x_1) = f(x_2)$

$\Rightarrow$ $2x_1 + 3 = 2x_2 + 3 \Rightarrow x_1 = x_2$

$\therefore$ f is one-one function.

Now, let $y = 2x + 3 \Rightarrow x = \frac{y-3}{2}$

It is not onto function, because for $y = 3$, $x = 0 \notin N$.

Hence, it is one-one into function.

35. ***(d)*** Let $y = x^2 - 6x + 7$

$\Rightarrow$ $x^2 - 6x + 7 - y = 0$

On comparing with $ax^2 + bx + c = 0$, we get

$a = 1, b = -6$ and $c = (7 - y)$

Now, using Sridharacharya formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{a}$

$$= \frac{6 \pm \sqrt{36 - 4(7-y)}}{2}$$

$$= \frac{6 \pm \sqrt{36 - 28 + 4y}}{2} = \frac{6 \pm \sqrt{4y + 8}}{2}$$

$$= \frac{6 \pm 2\sqrt{y+2}}{2} = 3 \pm \sqrt{y+2}$$

$f(x)$ is defined only when

$y + 2 \geq 0 \Rightarrow y \geq -2$

$\therefore$ Range of $f(x) = [-2, \infty)$

36. ***(b)*** Let α be the angle between two forces P and Q. It is given that the resultant of P and Q is of magnitude P.

Therefore, $P^2 = P^2 + Q^2 + 2PQ\cos\alpha$

$\Rightarrow$ $Q(Q + 2P\cos\alpha) = 0$

$\Rightarrow$ $Q + 2P\cos\alpha = 0$ $\quad \ldots(\text{i})$

$2P\cos\alpha = -Q$

Let θ be the angle between the force Q and new resultant.

Then, $\tan\theta = \frac{2P\sin\alpha}{Q + 2P\cos\alpha}$

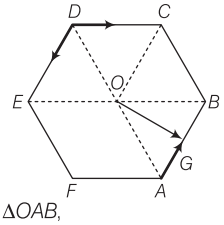
$\Rightarrow$ $\tan\theta = \frac{2P\sin\alpha}{Q - Q} = \infty$ [from Eq. (i)]

$\Rightarrow$ $\theta = \frac{\pi}{2}$

37. ***(d)*** Let A be the area of cross section of the rod.

Then, $\bar{x} = \dfrac{\int_0^L x \cdot kx^2 a\,dx}{\int_0^L kx^2 \cdot a\,dx} = \dfrac{\int_0^L x^3 dx}{\int_0^L x^2 dx} = \dfrac{\frac{L^4}{4}}{\frac{L^3}{3}} = \dfrac{3}{4}L$

38. ***(d)*** We know that, moment force = rF



In right angled ΔOAB,

$$OG^2 = OA^2 - AG^2$$

$\Rightarrow$ $OG^2 = a^2 - \left(\dfrac{a}{2}\right)^2$ [if OAB is an equilateral triangle with side a, then $AG = \dfrac{AB}{2} = \dfrac{a}{2}$]

$\Rightarrow$ $OG^2 = \dfrac{3}{4}a^2 \Rightarrow OG = \dfrac{\sqrt{3}a}{2}$

Moment of force AB about 'O' is (OG) **AB** i.e. $\dfrac{\sqrt{3}}{2}a(F)$

Moment of force DC about 'O' is (OG) **CD**, i.e. $\dfrac{\sqrt{3}}{2}a\,(-F)$

Moment of force DE about 'O' is (OG) **DE**, i.e. $\dfrac{\sqrt{3}}{2}a\,(F)$

$\therefore$ Resultant moment of force about 'O'

$$= \left(\frac{\sqrt{3}}{2}a\right)F - \left(\frac{\sqrt{3}}{2}a\right)F + \left(\frac{\sqrt{3}}{2}a\right)F = \left(\frac{\sqrt{3}}{2}a\right)F$$

39. ***(c)*** Clearly, the position vector of particle at any time t is

$$\mathbf{r} = a(t + \sin t)\hat{\mathbf{i}} + a(1 - \cos t)\hat{\mathbf{j}}$$

Now, $\dfrac{d\mathbf{r}}{dt} = a(1 + \cos t)\,\hat{\mathbf{i}} + a\sin t\hat{\mathbf{j}}$

$\therefore$ $\dfrac{d^2\mathbf{r}}{dt^2} = -a\sin t\hat{\mathbf{i}} + a\cos t\,\hat{\mathbf{j}}$

Hence, required acceleration at any time t

$$= -a\sin t\,\hat{\mathbf{i}} + a\cos t\,\hat{\mathbf{j}}$$

and magnitude of acceleration

$$= \sqrt{(-a\sin t)^2 + (a\cos t)^2} = \sqrt{a^2\sin^2 t + a^2\cos^2 t}$$

$$= \sqrt{a^2(\sin^2 t + \cos^2 t)} = \sqrt{a^2 \times 1} = a$$

40. ***(d)*** Given, $x = \sqrt{t}$

On differentiating both sides w.r.t. t, we get

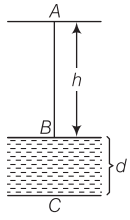
$$\frac{dx}{dt} = \frac{1}{2\sqrt{t}} = v \text{ (say)}$$

Again differentiating both sides w.r.t. t, we get

$$\frac{d^2x}{dt^2} = \frac{1}{2}\left(-\frac{1}{2}t^{-3/2}\right) = -\frac{1}{4t^{3/2}} = a \text{ (say)}$$

Now, $\dfrac{a}{v^3} = \dfrac{-\frac{1}{4t^{3/2}}}{\frac{1}{8t^{3/2}}} = -\dfrac{1}{4} \times 8 = -2$

41. ***(c)*** Here, $AB = h$ and $BC = d$



Velocity at point B,

$v_0^2 = u^2 + 2gh$

$\Rightarrow$ $v_0^2 = 0 + 2gh$ [$\because u = 0$]

$\Rightarrow$ $v_0^2 = 2gh$ ...(i)

Since, the body comes to rest at point C.

Then, $v = 0$

Now, $v^2 = v_0^2 - 2ad$ [where, a is deceleration]

$\Rightarrow$ $0 = v_0^2 - 2ad$

$\Rightarrow$ $2ad = v_0^2$

$\Rightarrow$ $2ad = 2gh$ [$\because$ at point C, $u^2 = 2gh$]

$\Rightarrow$ $a = \dfrac{gh}{d}$

42. ***(a)*** Given equation of parabola is

$$y^2 = 16x$$

On differentiating both sides, we get

$$2yy' = 16 \Rightarrow y' = \frac{16}{2y} = \frac{8}{y}$$

$\therefore$ Slope of tangent at point (x_1, y_1), $m_1 = \dfrac{8}{y_1}$

and slope of normal at point (x_1, y_1), $m_2 = \dfrac{-y_1}{8}$

Since, normal makes equal angle with both X and Y-axes, then

$m_2 = \pm 1$

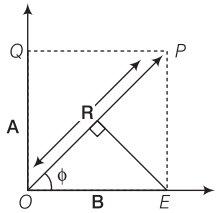
$\therefore$ $\dfrac{-y_1}{8} = \pm 1 \Rightarrow -y_1 = \pm 8$

Now, when $y_1 = 8$, then $x_1 = 4$

and when $y_1 = -8$, then $x_1 = 4$

So, the required point is $(4, -8)$.

43. ***(c)*** Here, $|\mathbf{A}| = 3, |\mathbf{B}| = 4$



Let $\angle EOP = \phi$, then $\mathbf{B} = R\cos\phi = 4$ and $\mathbf{A} = R\sin\phi = 3$

$\therefore$ $\tan\phi = \dfrac{3}{4}$

Now, $\cos\phi = \dfrac{4}{5}$

Projection of $\mathbf{B}$ on $OP = |\mathbf{B}|\cos\phi$

$$= 4\cos\phi = 4 \times \frac{4}{5} = \frac{16}{5} = 3.2$$

44. *(d)* Given, $\mathbf{R} = \mathbf{A} \times (\mathbf{B} \times \mathbf{C})$

Here, vector **R** is perpendicular to **A** and lies in the plane of **B** and **C**.

So, none of the options is true.

45. *(a)* Given, $\frac{dy}{dx} = 2e^{x-y} + x^2e^{-y}$

$$\Rightarrow \quad \frac{dy}{dx} = 2e^x \cdot \frac{1}{e^y} + x^2 \cdot \frac{1}{e^y} \quad \Rightarrow \frac{dy}{dx} = \frac{1}{e^y}(2e^x + x^2)$$

$$\Rightarrow \quad e^y dy = (2e^x + x^2)\,dx$$

On integrating both sides, we get

$$\int e^y dy = \int (2e^x + x^2)\,dx \quad \Rightarrow e^y = 2e^x + \frac{x^3}{3} + C$$

which is the required solution.

46. *(c)* Given, $(x + 2y^3)\frac{dy}{dx} = y \Rightarrow y\frac{dx}{dy} = x + 2y^3 \Rightarrow \frac{dx}{dy} - \frac{1}{y}x = 2y^2$

This is of the form $\frac{dx}{dy} + Px = Q$.

where, $P = -\frac{1}{y}$ and $Q = 2y^2$

Thus, the given equation is linear.

$$\therefore \text{IF} = e^{\int P dy} = e^{\int -\frac{1}{y}dy} = e^{-\log y} = e^{\log (y)^{-1}} = y^{-1} = \frac{1}{y}$$

So, the required solution is

$$x \cdot \text{IF} = \int (Q \cdot \text{IF})\,dy + C \quad \Rightarrow \quad x \cdot \frac{1}{y} = \int \left(2y^2 \cdot \frac{1}{y}\right) dy + C$$

$$\Rightarrow \quad x \cdot \frac{1}{y} = \int 2y\,dy + C = y^2 + C \quad \Rightarrow \quad x = y^3 + Cy$$

47. *(c)* $\lim\limits_{n \to \infty} \frac{1}{n^3}(1^2 + 2^2 + 3^2 + \ldots + n^2)$

$$= \lim_{n \to \infty} \frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] = \lim_{n \to \infty} \frac{\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)}{6}$$

$$= \frac{1 \times 2}{6} = \frac{1}{3}$$

48. *(a)* We have, $f(0) = a$

$$\therefore \quad \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} h \sin \frac{1}{h} = 0$$

and $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{h \to 0} f(0 - h) = \lim\limits_{h \to 0} (-h)\sin \frac{1}{-h} = 0$

Since, $f(x)$ is continuous at $x = 0$, so we must have

$$f(0) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) \Rightarrow a = 0$$

49. *(a)* Given, $y = x^{\sin x}$

On taking log both sides, we get

$$\log y = \sin x \log x$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = \sin x \frac{d}{dx}\log x + \log x \frac{d}{dx}\sin x$$

$$\Rightarrow \quad \frac{1}{y} \cdot \frac{dy}{dx} = \sin x \cdot \frac{1}{x} + \log x \cdot \cos x$$

$$\Rightarrow \quad \frac{dy}{dx} = y\left(\frac{\sin x}{x} + \log x \cdot \cos x\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = x^{\sin x}\left(\frac{\sin x}{x} + \log x \cos x\right)$$

50. *(b)* Given, $y = x^3 - 2x^2 + x - 2$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 3x^2 - 4x + 1 \text{ and } y = x \Rightarrow \frac{dy}{dx} = 1$$

$\therefore$ Slope of tangent will be $3x^2 - 4x + 1$.

Since, the tangent is parallel to line $y = x$.

$$\therefore \quad 3x^2 - 4x + 1 = 1$$

$$\Rightarrow \quad 3x^2 - 4x = 0 \Rightarrow x(3x - 4) = 0 \Rightarrow x = 0, \frac{4}{3}$$

When $x = 0$, then $y = -2$

When $x = \frac{4}{3}$, then $y = \frac{-50}{27}$

Now, equation of tangents at point $(0, -2)$ is

$$y - y_1 = \frac{dy}{dx}(x - x_1)$$

$$y + 2 = 1(x - 0)$$

$$\Rightarrow \quad y + 2 = x \quad \Rightarrow \quad x - y = 2 \qquad \ldots(i)$$

and equation of tangents at point $\left(\frac{4}{3}, -\frac{50}{27}\right)$ is

$$y - y_1 = \frac{dy}{dx}(x - x_1)$$

$$y + \frac{50}{27} = x - \frac{4}{3}$$

$$\Rightarrow \quad x - y = \frac{50}{27} + \frac{4}{3}$$

$$\Rightarrow \quad x - y = \frac{50 + 36}{27}$$

$$\Rightarrow \quad x - y = \frac{86}{27} \qquad \ldots(ii)$$

Hence, Eqs. (i) and (ii) are required equations of the tangents.

Notice Board

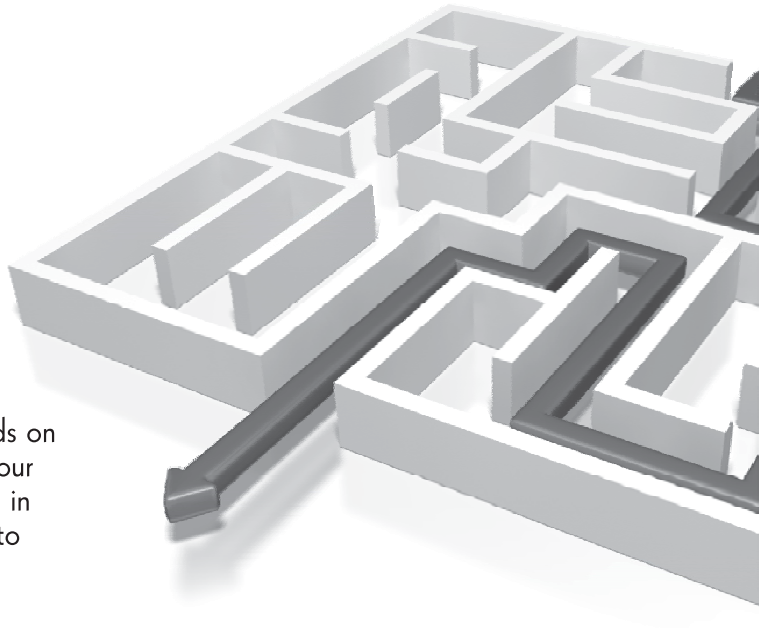
No weightage to class XII marks from 2017

HRD minister officials said that, class XII marks will not be a factor for determining rankings in the Joint Entrance Examination (JEE) for admissions to engineering undergraduate courses from next year.

Class XII scores will be a crucial parameter only for determining the eligibility criteria for the exam. To qualify for appearing in exam, students need to have atleast 75% marks or be in top 20 percentile in class XII examination. SC, ST students should have 65% marks.

Also, there will not be any change in the existing two-tier pattern of JEE Main and Advanced.

SHORTCUT METHODS



Here, we are introducing some important shortcut methods on quadratic equation and area of bounded region to save your precious time in tedious calculations based problem asked in various Engineering Entrance exams. Use these shortcuts, to practice questions and be acquainted with it.

Quadratic Equation

1 If $a > 0$ and $x = \sqrt{a + \sqrt{a + \ldots \infty}}$, then $x = \dfrac{1 + \sqrt{4a + 1}}{2}$

Example *The value of* $\sqrt{42 + \sqrt{42 + \sqrt{42 + \ldots}}}$ *is*

(a) 7 (b) –6 (c) 5 (d) 4

Sol. **(a)** Clearly,

$$\sqrt{42 + \sqrt{42 + \sqrt{42 + \ldots}}} = \frac{1 + \sqrt{168 + 1}}{2} = \frac{1 + \sqrt{169}}{2} = \frac{1 + 13}{2} = 7$$

2 In the equation $ax^2 + bx + c = 0$, $[a, b, c \in R]$, if $a + b + c = 0$, then the roots are $1, \dfrac{c}{a}$ and if $a - b + c = 0$, then the roots are $-1, \dfrac{-c}{a}$.

Example *Sum of squares of roots of the equation* $x^2 + x - 2 = 0$ *is*

(a) 0 (b) 4 (c) 5 (d) None of these

Sol. **(c)** Clearly, the roots of the given equation are 1 and –2. Hence, sum of squares of roots $= 1^2 + (-2)^2 = 5$

3 (i) If $D = b^2 - 4ac$ is a perfect square (of a rational number) and $a, b, c \in Q$, then the equation $ax^2 + bx + c = 0$ has rational roots.

(ii) If $D = b^2 - 4ac > 0$ and not a perfect square, then the equation $ax^2 + bx + c = 0$ has irrational roots.

Example *The roots of the quadratic equation* $2x^2 + 3x + 1 = 0$, *are*

(a) irrational (b) rational

(c) imaginary (d) None of these

Sol. **(b)** Here $D = 1$, which is a perfect square. Hence, its roots are rational.

4 For the quadratic equation $ax^2 + bx + c = 0$,

(i) one root will be reciprocal of the other, if $a = c$

(ii) one root is zero, if $c = 0$

(iii) roots are equal in magnitude but opposite in sign, if $b = 0$

(iv) both roots are zero, if $b = c = 0$

Example *If the roots of the equation*

$$\frac{1}{x + p} + \frac{1}{x + q} = \frac{1}{r}, (x \neq -p, x \neq -q, r \neq 0)$$

are equal in magnitude but opposite in sign, then $p + q$ *is equal to*

(a) r (b) $2r$ (c) r^2 (d) $\dfrac{1}{r}$

Sol. **(b)** The given equation can be rewritten as $x^2 + (p + q - 2r)x + pq - r(p + q) = 0$.

As we know, if roots are equal in magnitude but opposite in sign, then coefficient of x will be zero.

$\therefore \quad p + q - 2r = 0 \Rightarrow p + q = 2r$

5 Maximum and Minimum value of Quadratic Equation

(a) If $a > 0$, then minimum value of $ax^2 + bx + c$ is $\dfrac{4ac - b^2}{4a}$. But there is no greatest value.

(b) If $a < 0$, then maximum value of $ax^2 + bx + c$ is $\dfrac{4ac - b^2}{4a}$. But there is no least value.

Example *If* x *is real, then the maximum value of* $5 + 4x - 4x^2$ *will be*

(a) 5 (b) 6 (c) 1 (d) 2

Sol. ***(b)*** Here, $a = -4 < 0$, therefore maximum value of given expression is given by

$$\frac{4ac - b^2}{4a} = \frac{4(-4)(5) - (4)^2}{4(-4)} = \frac{-80 - 16}{-16} = \frac{-96}{-16} = 6$$

6 If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then the equation whose roots are

(i) $-\alpha, -\beta$ can be obtained by replacing x by $-x$

(ii) $\alpha + k, \beta + k$ can be obtained by replacing x by $x - k$

(iii) $\alpha - k, \beta - k$ can be obtained by replacing x by $x + k$

(iv) $\frac{1}{\alpha}, \frac{1}{\beta}$ can be obtained by replacing x by $\frac{1}{x}$ or just change the coefficients a, b, c to c, b, a.

(v) $\frac{k}{\alpha}, \frac{k}{\beta}$ can be obtained by replacing x by $\frac{k}{x}$

(vi) $\alpha k, \beta k$ can be obtained by replacing x by $\frac{x}{k}$

(vii) $\frac{\alpha}{k}, \frac{\beta}{k}$ can be obtained by replacing x by xk

(viii) α^2, β^2 is $a^2x^2 - (b^2 - 2ac)x + c^2 = 0$

(ix) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ is $x^2 - \left(\frac{b^2 - 2ac}{ac}\right)x + 1 = 0$

Example *The equation whose roots are reciprocal, of the roots of the equations* $3x^2 - 20x + 17 = 0$ *is*

(a) $3x^2 + 20x - 17 = 0$ *(b)* $17x^2 - 20x + 3 = 0$

(c) $17x^2 + 20x + 3 = 0$ *(d)* None of these

Sol. ***(b)*** On replacing x by $\frac{1}{x}$ in the given equation, we get

$$3\left(\frac{1}{x^2}\right) - 20\left(\frac{1}{x}\right) + 17 = 0$$

$$\Rightarrow \quad 17x^2 - 20x + 3 = 0$$

Example *The quadratic equation whose roots are three times the roots of* $3ax^2 + 3bx + c = 0$, *is*

(a) $ax^2 + 3bx + 3c = 0$ *(b)* $ax^2 + 3bx + c = 0$

(c) $9ax^2 + 9bx + c = 0$ *(d)* $ax^2 + bx + 3c = 0$

Sol. ***(a)*** On replacing x by $\frac{x}{3}$ in the given equation , we get

$$3a\left(\frac{x}{3}\right)^2 + 3b\left(\frac{x}{3}\right) + c = 0$$

$$\Rightarrow \quad a\frac{x^2}{3} + bx + c = 0$$

$$\Rightarrow \quad ax^2 + 3bx + 3c = 0$$

7 If α is repeated root of the quadratic equation $ax^2 + bx + c = 0$, then $\alpha = \frac{-b}{2a}$.

Example *If* α *is repeated root of the equation* $x^2 - 4x - 2\sqrt{3}x + 7 + 4\sqrt{3} = 0$, *then* α *is equal to*

(a) $\sqrt{3}$ *(b)* $2\sqrt{3}$ *(c)* $2 + \sqrt{3}$ *(d)* $2 - \sqrt{3}$

Sol. ***(c)*** On comparing the given equation with $ax^2 + bx + c = 0$, we get $a = 1$ and $b = -(4 + 2\sqrt{3})$

Hence, $\alpha = \frac{4 + 2\sqrt{3}}{2} = 2 + \sqrt{3}$

8 If all the terms of an equation are positive and the equation involves no odd power of x, then its all roots are complex.

Note The roots of the equation $x^2 + x + 1 = 0$ are ω and ω^2, where $\omega = \frac{-1}{2} + i\frac{\sqrt{3}}{2}$ and $\omega^2 = \frac{-1 - i\sqrt{3}}{2}$.

Example *If* α, β *are the roots of the quadratic equation* $x^2 + x + 1 = 0$, *then the equation whose roots are* α^{19}, β^7, *is*

(a) $x^2 - x + 1 = 0$ *(b)* $x^2 - x - 1 = 0$

(c) $x^2 + x - 1 = 0$ *(d)* $x^2 + x + 1 = 0$

Sol. ***(d)*** Clearly, the roots of $x^2 + x + 1 = 0$ are ω and ω^2.

Let $\alpha = \omega$ and $\beta = \omega^2$

Then, $\alpha^{19} = \omega^{18} \cdot \omega$ and $\beta^7 = \omega^{12} \cdot \omega^2$

$\Rightarrow$ $\alpha^{19} = \omega$ and $\beta^7 = \omega^2$

$\therefore$ Required equation is same as the given equation.

9 To find the common root of two quadratic equations, make the coefficient of second degree term in the two equations equal and subtract. The value of x so obtained is the required common root.

Example *The common root of the equations* $x^2 - x - 12 = 0$ *and* $3x^2 + 10x + 3 = 0$ *is*

(a) 3 *(b)* –3 *(c)* 4 *(d)* –4

Sol. ***(b)*** Given equations can be written as

$$x^2 - x - 12 = 0 \quad \ldots(i)$$

$$x^2 + \frac{10}{3}x + 1 = 0 \quad \ldots(ii)$$

On subtracting Eq. (ii) from Eq. (i), we get

$$\left(-1 - \frac{10}{3}\right)x - 12 - 1 = 0 \Rightarrow \frac{-13}{3}x = 13 \Rightarrow x = -3$$

10 (i) If $ax^2 + bx + c < 0, (a > 0)$, then $x \in (\alpha, \beta)$, where $\alpha, \beta (\alpha < \beta)$ are roots of the equation $ax^2 + bx + c = 0$

(ii) If $ax^2 + bx + c > 0, (a > 0)$, then $x \in (-\infty, \alpha) \cup (\beta, \infty)$, where $\alpha, \beta\ (\alpha < \beta)$ are roots of the equation $ax^2 + bx + c = 0$.

Example *The complete solution of the inequation* $x^2 - 4x < 12$ *is*

(a) $x < -2$ or $x > 6$ *(b)* $-6 < x < 2$

(c) $2 < x < 6$ *(d)* $-2 < x < 6$

Sol. ***(d)*** We have, $x^2 - 4x < 12 \Rightarrow x^2 - 4x - 12 < 0$

$\Rightarrow$ $(x - 6)(x + 2) < 0$

$\Rightarrow$ $-2 < x < 6$

Area of Bounded Region

1 Area of the smaller region bounded by a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ and its chord $y = mx + c'$ such that chord subtends a right angle at its centre is $\frac{(\pi - 2)\, r^2}{4}$ sq unit.

Example *Smaller area enclosed by the circle* $x^2 + y^2 = 4$ *and the line* $x + y = 2$ *is (in sq unit)*
(a) $2(\pi - 2)$ *(b)* $(\pi - 2)$ *(c)* $2\pi - 1$ *(d)* $2(\pi + 2)$

Sol. ***(b)*** Here, $r = 2$

$\therefore$ Required area $= \frac{(\pi - 2)\, 4}{4} = \pi - 2$ sq unit

2 The area of the larger region bounded by a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ and its chord $y = mx + c$ such that the chord subtends a right angle at its centre is $\frac{(3\pi + 2)\, r^2}{4}$ sq unit.

Example *Larger area enclosed by the circle* $x^2 + y^2 = 9$ *and the line* $x + y = 3$ *is (in sq unit)*
(a) $(3\pi + 2)$ *(b)* $\frac{9}{4}(3\pi + 2)$ *(c)* $\frac{3\pi + 2}{4}$ *(d)* None

Sol. ***(b)*** Here, $r = 3$

$\therefore$ Required area $= \frac{(3\pi + 2)\, 9}{4}$ sq unit

3 Area of the region enclosed by two intersecting circles $c_1 : x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $c_2 : x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ is

$$r_1^2 \sin^{-1}\left(\frac{p}{r_1}\right) - pd + r_2^2 \sin^{-1}\left(\frac{p}{r_2}\right) - ps$$

Here, d is distance of common chord from centre of first circle and r_1 is its radius. While s is distance of common chord from centre of second circle and r_2 is its radius. $2p$ is length of common chord.

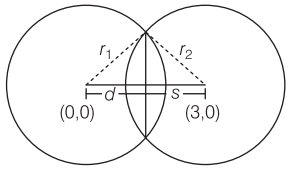
Example *Area of the region enclosed between the circles* $x^2 + y^2 = 9$ *and* $(x - 3)^2 + y^2 = 9$ *is (in sq units)*
(a) $\frac{12\pi - 9\sqrt{3}}{2}$ *(b)* $\frac{\pi + 9\sqrt{3}}{2}$ *(c)* $\frac{3\pi + 9\sqrt{3}}{2}$ *(d)* $\frac{3\pi + \sqrt{3}}{2}$

Sol. ***(a)*** Here, $r_1 = r_2 = 3$. Equation of common chord is

$s_1 - s_2 = 0 \Rightarrow 2x - 3 = 0$

$$d = \left|\frac{2(0) - 3}{\sqrt{2^2 + 0^2}}\right| = \frac{3}{2}$$

Also, $s = \left|\frac{2(3) - 3}{\sqrt{2^2 + 0^2}}\right| = \frac{3}{2}$



Thus, $p = \sqrt{3^2 - \left(\frac{3}{2}\right)^2} = \frac{3\sqrt{3}}{2}$

Now, area of the required region

$$= 9\sin^{-1}\left(\frac{3\sqrt{3}}{2 \cdot 3}\right) - \frac{3\sqrt{3}}{2} \times \frac{3}{2} + 9\sin^{-1}\left(\frac{3\sqrt{3}}{2 \cdot 3}\right) - \frac{3\sqrt{3}}{2} \times \frac{3}{2}$$

$$= 9\frac{\pi}{3} - \frac{9}{4}\sqrt{3} + 9\left(\frac{\pi}{3}\right) - \frac{9\sqrt{3}}{4} = 6\pi - \frac{9}{2}\sqrt{3} = \frac{12\pi - 9\sqrt{3}}{2} \text{ sq units}$$

4 The area enclosed by the graph $|x| + |y| = a$ is $2a^2$ sq unit.

Example *Area bounded by* $|x| + |y| = 4$ *is (in sq units)*
(a) 32 *(b)* 4 *(c)* 16 *(d)* 48

Sol. ***(a)*** Here, $a = 4$

Hence, required area bounded by the region

$= 2a^2 = 2(4)^2 = 32$ sq units

5 The area of the region bounded by $y^2 = 4ax$ and $x^2 = 4ay$ and $x = a$ is $\frac{5a^2}{4}$ sq units.

Example *Curve* $y^2 = 4x$, $x^2 = 4y$ *and* $x = 1$ *bounded the area*
(a) $\frac{2}{4}$ sq unit *(b)* $\frac{1}{2}$ sq unit *(c)* $\frac{5}{2}$ sq unit *(d)* $\frac{5}{4}$ sq unit

Sol. ***(d)*** Here, $a = 1$

$\therefore$ Required area $= \frac{5(1)^2}{4} = \frac{5}{4}$ sq unit

6 Area bounded by $y^2 = 4a(x + a)$ and $y^2 = 4b(b - x)$ is $\frac{8}{\sqrt{3}}\sqrt{ab}\,(a + b)$ sq unit.

Example *Area bounded by the curves* $y^2 = 4(x + 1)$ *and* $y^2 = 4(1 - x)$ *is (in sq unit)*
(a) $\frac{8}{\sqrt{3}}$ *(b)* $\frac{16}{\sqrt{3}}$ *(c)* $\frac{4}{\sqrt{3}}$ *(d)* $\frac{12}{\sqrt{3}}$

Sol. ***(b)*** Here, $a = 1, b = 1$

$\therefore$ Required area $= \frac{8}{3}\sqrt{1 \times 1}\,(1 + 1) = \frac{16}{\sqrt{3}}$ sq unit

7 The area of the region bounded by three straight lines $y = m_1x + c_1$, $y = m_2x + c_2$ and $y = m_3x + c_3$ is

$$\frac{1}{2}\left|\left[\frac{(c_1 - c_2)^2}{m_1 - m_2} + \frac{(c_2 - c_3)^2}{m_2 - m_3} + \frac{(c_3 - c_1)^2}{m_3 - m_1}\right]\right|.$$

Example *Area bounded by the lines* $y = 2x + 1$, $y = -3x + 2$ *and* $y = 4x + 3$ *is*
(a) $\frac{1}{35}$ sq unit *(b)* $\frac{36}{35}$ sq unit *(c)* 35 sq unit *(d)* 36 sq unit

Sol. ***(b)*** Here, $m_1 = 2, m_2 = -3$ and $m_3 = 4$

$c_1 = 1, \; c_2 = 2, \; c_3 = 3$

$$\therefore \text{ Required area} = \frac{1}{2}\left|\frac{(1 - 2)^2}{2 + 3} + \frac{(2 - 3)^2}{-3 - 4} + \frac{(3 - 1)^2}{4 - 2}\right|$$

$$= \frac{1}{2}\left|\frac{1}{5} - \frac{1}{7} + \frac{4}{2}\right| = \frac{1}{2}\left|\frac{14 - 10 + 140}{70}\right| = \frac{1}{2}\left(\frac{144}{70}\right) = \frac{36}{35} \text{ sq unit}$$

MINORS AND COFACTORS

Minor Let $A=(a_{ij})_{n\times n}$ be a square matrix of order n.

Then, the minor M_{ij} of the element a_{ij} of the matrix A is the determinant of the square submatrix of order $(n-1)$ obtained by deleting ith row and jth column of the matrix A.

Cofactor Let $A=(a_{ij})_{n\times n}$ be a square matrix of order n. Then, the cofactor of the element a_{ij} of the matrix A is denoted by C_{ij} and is equal to $(-1)^{i+j}M_{ij}$, where M_{ij} is the minor of the element a_{ij} of the matrix A.

ADJOINT OF A SQUARE MATRIX

Let $A=[a_{ij}]$ be a square matrix of order n and C_{ij} be cofactor of a_{ij}. Then, the adjoint of A, denoted by adj(A), is defined as the transpose of the cofactor matrix. The adjoint of a square matrix A is obtained by replacing each (i, j)th element of A by the cofactor of (j, i) th element in A.

PROPERTIES OF ADJOINT OF A SQUARE MATRIX

Let A be a square matrix of order n

(i) $A(\text{adj } A)=|A|I_n=(\text{adj } A)A$

(ii) $|\text{adj } A|=|A|^{n-1}$, if $|A|\neq 0$

(iii) $\text{adj}(\text{adj } A)=|A|^{n-2}\cdot A$, if $|A|\neq 0$

(iv) $|\text{adj}(\text{adj } A)|=|A|^{(n-2)^2}$, if $|A|\neq 0$

(v) $\text{adj}(A^T)=(\text{adj } A)^T$

(vi) $\text{adj}(A^m)=(\text{adj } A)^m \quad m\in N$

(vii) $\text{adj}(kA)=k^{n-1}(\text{adj } A), k\in R$

INVERSE OF A SQUARE MATRIX

The inverse of a square matrix A, denoted by A^{-1}, is determined by the formula $A^{-1}=(1/|A|)\,\text{adj}(A)$, provided $|A|\neq 0$

PROPERTIES OF INVERSE OF A SQUARE MATRIX

(i) $(A^{-1})^{-1}=A$ (ii) $AA^{-1}=A^{-1}A=I$

(iii) $(A^T)^{-1}=(A^{-1})^T$ (iv) $(AB)^{-1}=B^{-1}A^{-1}$

(v) $(ABC)^{-1}=C^{-1}B^{-1}A^{-1}$ (vi) $\text{adj}(A^{-1})=(\text{adj } A)^{-1}$

(vii) If $A=\text{diag}(a_1, a_2, a_3, \ldots, a_n)$, then $A^{-1}=\text{diag}(a_1^{-1}, a_2^{-1}, \ldots, a_n^{-1})$

(viii) $|A^{-1}|=|A|^{-1}$

RANK OF MATRIX

A positive integer r is said to be the rank of a non-zero matrix A, if

(i) there exists atleast one minor in A of order r, which is not zero.

(ii) every minor in A of order greater than r is zero. If rank of a matrix A is r, then it is written as $\rho(A)=r$.

PROPERTIES OF RANK OF A MATRIX

(i) The rank of a zero matrix (null matrix) is defined as zero, i.e. $\rho(O)=0$

(ii) If I_n is unit matrix of order n. Then, rank of I_n is n, i.e. $\rho(I_n)=n$.

(iii) If A is $n\times n$ non-singular matrix, then $\rho(A)=n$.

(iv) Elementary transformations does not change the rank of a matrix.

ECHELON FORM OF A MATRIX

A matrix A is said to be in Echelon form, if either A is the null matrix or A satisfies the following conditions

(i) All the non-zero rows in A precede the zero rows.

(ii) The number of zeroes preceding the first non-zero element in a row is less than number of such zeroes in the succeeding row.

(iii) The first non-zero element in a row is unity.

Note The number of non-zero rows of a matrix given in the Echelon form is its rank.

SOLUTION OF A SYSTEM OF LINEAR EQUATIONS BY MATRIX METHOD

NON-HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS

Consider a system of n linear equations in n variables.

$$\begin{aligned}
a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n&=b_1\\
a_{21}x_1+a_{22}x_2+\ldots+a_{2n}x_n&=b_2\\
\vdots\quad&\quad\vdots\quad\quad\vdots\quad\vdots\\
a_{n1}x_1+a_{n2}x_2+\ldots+a_{nn}x_n&=b_n
\end{aligned}$$

We can write the above system of equations in matrix form as $AX=B$, where $A=$ coefficient matrix,

$$X=\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix} \text{ and } B=\begin{bmatrix}b_1\\b_2\\\vdots\\b_n\end{bmatrix}$$

Now, the following cases may arise

(i) If $|A|\neq 0$, then the system of equations is consistent and has a unique solution given by $X=A^{-1}B$.

(ii) If $|A|=0$ and $(\text{adj } A)\,B\neq O$, then system of equations is inconsistent, i.e. it has no solution.

(iii) If $|A|=0$ and $(\text{adj } A)\,B=O$, then the system of equations may be either consistent or inconsistent. However, if the system of equations is consistent, then it will have infinite number of solutions.

HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS

The system of equations $AX=B$ is said to be homogeneous, if the matrix B is a zero matrix. In this case system is of the form $AX=O$, where O is a null matrix and its solution is given by the following conditions

(i) If $|A|\neq 0$, then its only solution is $X=O$, which is called the trivial or zero solution.

(ii) If $|A|=0$, then $AX=O$ have non-trivial solutions and it will be infinite in numbers.

SOLUTION OF SYSTEM OF LINEAR EQUATIONS BY DETERMINANTS (OR CRAMER'S RULE)

FOR NON-HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS

Consider, the system of equations

$a_1x+b_1y+c_1z=d_1$, $a_2x+b_2y+c_2z=d_2$ and $a_3x+b_3y+c_3z=d_3$. Let

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DETERMINANTS AND ITS PROPERTIES

DETERMINANTS

Every square matrix $A=[a_{ij}]$ of order n can be associated to a number (real or complex), which is called the determinant of square matrix A.

SECOND ORDER DETERMINANT

A determinant of order two is of the form, $\begin{vmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{vmatrix}$

and is equal to $a_{11}a_{22}-a_{12}a_{21}$.

THIRD ORDER DETERMINANT

A determinant of order three is of the form, $\begin{vmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{vmatrix}$ and is equal to

$$a_{11}\begin{vmatrix}a_{22}&a_{23}\\a_{32}&a_{33}\end{vmatrix}-a_{12}\begin{vmatrix}a_{21}&a_{23}\\a_{31}&a_{33}\end{vmatrix}+a_{13}\begin{vmatrix}a_{21}&a_{22}\\a_{31}&a_{32}\end{vmatrix}$$

$$=a_{11}(a_{22}a_{33}-a_{23}a_{32})-a_{12}(a_{21}a_{33}-a_{23}a_{31})+a_{13}(a_{21}a_{32}-a_{31}a_{22}).$$

PROPERTIES OF DETERMINANT

(i) The value of determinant remains unchanged, if rows are changed into columns and columns are changed into rows. Thus $|A|=|A^T|$.

(ii) If any two rows (or columns) of a determinant are interchanged, then the value of determinant so obtained is the negative of the value of the original determinant.

(iii) If two rows (or columns) of a determinant are identical, then the value of determinant is zero.

(iv) If any two rows (or columns) of a determinant are proportional, then its value is zero.

$$D=\begin{vmatrix}a_1&b_1&c_1\\a_2&b_2&c_2\\a_3&b_3&c_3\end{vmatrix}, D_1=\begin{vmatrix}d_1&b_1&c_1\\d_2&b_2&c_2\\d_3&b_3&c_3\end{vmatrix}$$

$$D_2=\begin{vmatrix}a_1&d_1&c_1\\a_2&d_2&c_2\\a_3&d_3&c_3\end{vmatrix} \text{ and } D_3=\begin{vmatrix}a_1&b_1&d_1\\a_2&b_2&d_2\\a_3&b_3&d_3\end{vmatrix}$$

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inants

(v) If each element of a row (or a column) of a determinant is zero, then the value of determinant is zero.

(vi) If all the elements of a determinant above or below the main diagonal consists of zeroes, then the value of the determinant is equal to the product of diagonal elements.

(vii) If each element of a row (or a column) of a determinant is multiplied by a constant k, then the value of the new determinant is k times the value of original determinant.

(viii) If each element of a row (or a column) of a determinant is the sum of two or more terms, then the determinant can be expressed as the sum of two or more determinants.

(ix) If each element of a row (or a column) of a determinant is multiplied by a constant k and then added to the corresponding elements of some other row (or column), the value of determinant remains same.

(x) The sum of product of elements of any row (or column) of a determinant with the cofactors of the corresponding elements of any other row (or column) is zero.

(xi) The sum of product of elements of any row (or column) of a determinant with their cofactors gives the value of determinant.

(xii) If the elements of a determinant are polynomial in x and if it vanishes for $x = a$, then $(x - a)$ must be factor of determinant. In other words, if two rows (or two columns) become identical for $x = a$, then $(x - a)$ is a factor of determinant.

In general, we can say, if r rows (or r columns) become identical when a is substituted for x, then $(x-a)^{r-1}$ is factor of given determinant.

Now, the following cases may arise

(i) If $D \neq 0$, then the system is consistent and has a unique solution, which is given by $x = D_1 / D$, $y = D_2 / D$ and $z = D_3 / D$.

(ii) If $D = 0$ and atleast one of the determinants D_1, D_2 and D_3 is non-zero. Then, the given system is inconsistent, i.e. it has no solution.

(iii) If $D = 0$ and $D_1 = D_2 = D_3 = 0$, then the system may or may not be consistent. However, if it is consistent, then it has infinitely many solutions.

FOR HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS

If the system of equation is homogeneous, then $D_1 = D_2 = D_3 = 0$ (value of determinant is zero, because one column has all elements = 0). Now, the solution is given by following conditions

(i) If $D \neq 0$, then system has only trivial solution, i.e. $x = y = z = 0$.

(ii) If $D = 0$, then the system has infinitely many solutions.

SOLUTION OF SYSTEM OF LINEAR EQUATIONS BY RANK METHOD

FOR NON-HOMOGENEOUS SYSTEM OF EQUATIONS

Let $AX = B$ be a system of n linear equations in n variables.

***Step* I** Write the augmented matrix $[A:B]$.

***Step* II** Reduce the augmented matrix to Echelon form using elementary row (column) operations.

***Step* III** Determine the rank of coefficient matrix A and augmented matrix $[A:B]$ by counting the number of non-zero rows in Echelon form of A and $[A:B]$.

***Step* IV**

(i) If $\rho(A) \neq \rho[A:B]$, then the system of equations is inconsistent.

(ii) If $\rho(A) = \rho[A:B]$ = number of unknowns. Then, the system of equation is consistent and has a unique solution.

(iii) If $\rho(A) = \rho[A:B] <$ number of unknowns. Then, the system of equations is consistent and has infinitely many solutions.

FOR HOMOGENEOUS SYSTEM OF EQUATIONS

Let $AX = O$ be the homogeneous system of linear equations.

(a) If $\rho(A)$ = number of variables, then $AX = O$ has trivial solution only, i.e. $X = 0$.

(b) If $\rho(A) <$ number of variables, then $AX = O$ has non-trivial solutions and it will be infinite in numbers.

CHARACTERISTIC EQUATION OF A MATRIX

The equation $|A - \lambda I_n| = 0$ is called characteristic equation of the matrix $A = [a_{ij}]_{n \times n}$. The values of λ are called eigen values of A.

PROPERTIES OF CHARACTERISTIC EQUATION

(i) The sum of eigen values of A is equal to its trace.

(ii) The product of eigen values of matrix A is equal to its determinant.

(iii) The eigen values of triangular matrix are its diagonal elements.

(iv) If λ is an eigen value of A and $|A| \neq 0$, then

(a) $\frac{1}{\lambda}$ is eigen value of A^{-1}.

(b) $\frac{|A|}{\lambda}$ is eigen value of $\text{adj}(A)$.

(v) If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of A, then eigen values of A^2 are $\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2$.

DIFFERENTIATION OF DETERMINANTS

Let $\Delta(x) = \begin{vmatrix} f_1(x) & g_1(x) \\ f_2(x) & g_2(x) \end{vmatrix}$, where $f_1(x), f_2(x), g_1(x)$ and $g_2(x)$ are functions of x.

Then, $\frac{d}{dx}\Delta(x) = \Delta'(x) = \begin{vmatrix} f_1'(x) & g_1'(x) \\ f_2(x) & g_2(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & g_1(x) \\ f_2'(x) & g_2'(x) \end{vmatrix}$

Also, $\Delta'(x) = \begin{vmatrix} f_1'(x) & g_1(x) \\ f_2'(x) & g_2(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & g_1'(x) \\ f_2(x) & g_2'(x) \end{vmatrix}$

Thus, to differentiate a determinant, we differentiate one row or column at a time, keeping others unchanged.

INTEGRATION OF DETERMINANTS

If $\Delta(x) = \begin{vmatrix} f(x) & g(x) \\ \lambda_1 & \lambda_2 \end{vmatrix}$,

then $\int_a^b \Delta(x)\,dx = \begin{vmatrix} \int_a^b f(x)\,dx & \int_a^b g(x)\,dx \\ \lambda_1 & \lambda_2 \end{vmatrix}$

Here, $f(x)$ and $g(x)$ are functions of x and λ_1, λ_2 are constants.

MAXIMUM AND MINIMUM VALUES OF DETERMINANTS WHEN ELEMENTS ARE KNOWN

If $|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix}$, where $a_i's \in \{\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n\}$

Then, $|A|_{\max}$ when diagonal elements are $\min(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)$ and non-diagonal elements are $\max(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)$.

Also, $|A|_{\min} = -|A|_{\max}$

PRODUCT OF DETERMINANTS OF SAME ORDER

Let $\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$

Then, row-by-row multiplication of Δ_1 and Δ_2 is given by

$$\Delta_1\Delta_2 = \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}$$

Note Multiplication can also be performed row by column, column by row or column by column as required in the problem.

Jitender Gupta

SOLUTION OF SYSTEM OF SIMULTANEOUS LINEAR EQUATIONS

Solution of system of simultaneous linear equations in two or three variables can be solved by using matrix method, rank method and determinant. We can also determine the nature of solution, i.e. whether the system of equation possesses unique solution, infinite solution or no solution. Here, we will provide article which will build up the basic concept regarding various engineering exams specially JEE.

System of Simultaneous Linear Equations

Consider the following system of n linear equations in n unknowns as given below

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n$$

Above system of equations can be written in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & \ldots & \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}_{n\times n} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}_{n\times 1} \quad \text{or} \quad AX = B$$

Then, $n \times n$ matrix A is called the coefficient matrix of the system of linear equations.

Homogeneous and Non-homogeneous System of Linear Equations

A system of equations $AX = B$ is called a homogeneous system, if $B = 0$. Otherwise, it is called a non-homogeneous system of equations.

Consistent System

If the system of equations has one or more solutions, then it is said to be consistent, otherwise it is inconsistent.

Solution of a Non-homogeneous System of Simultaneous Linear Equations

There are three methods for solving a non-homogeneous system of simultaneous linear equations.

Method I

This method is known as Cramer's rule or determinant method.

System of Two Linear Equations in Two Variables

Let the given system of equation be

$$a_1x + b_1y = d_1, \quad a_2x + b_2y = d_2$$

Let $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, $\Delta_1 = \begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}$

Then, the solution is given as $x = \frac{\Delta_1}{\Delta}$, $y = \frac{\Delta_2}{\Delta}$

Example 1 Consider the following system of linear equations $2x + 3y = 5$ and $x + 2y = 7$. Then,

(a) $x = 9, y = -11$ (b) $x = -11, y = 9$
(c) $x = y = 9$ (d) $x = y = -11$

Sol. **(b)** PROBLEM SOLVING STRATEGY Firstly, find $\Delta, \Delta_1, \Delta_2$ and then use the formula $x = \frac{\Delta_1}{\Delta}$, $y = \frac{\Delta_2}{\Delta}$.

We have, $2x + 3y = 5$; $x + 2y = 7$

$$\therefore \quad \Delta = \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 4 - 3 = 1$$

$$\Delta_1 = \begin{vmatrix} 5 & 3 \\ 7 & 2 \end{vmatrix} = 10 - 21 = -11$$

$$\Delta_2 = \begin{vmatrix} 2 & 5 \\ 1 & 7 \end{vmatrix} = 14 - 5 = 9$$

$$\therefore \quad x = \frac{\Delta_1}{\Delta} = \frac{-11}{1} = -11 \text{ and } y = \frac{\Delta_2}{\Delta} = \frac{9}{1} = 9$$

Nature of Solution

(i) If $\Delta \neq 0$, then system has unique solution and hence system of equations will be consistent.

(ii) If $\Delta = 0$ and atleast one of Δ_1, Δ_2 is non-zero, then system has no solution and hence system of equations will be inconsistent.

(iii) If $\Delta = \Delta_1 = \Delta_2 = 0$, then system has infinite solutions and hence system of equation will be consistent.

Example 2 If the system of equations $a^2x - by = a^2 - b$ and $bx - b^2y = 2 + 4b$ possess unique solution, then

(a) $a \neq \pm 1$ (b) $b \neq 0$
(c) Both (a) and (b) (d) None of these

Sol. **(c)** PROBLEM SOLVING STRATEGY For a system to possess a unique solution, $\Delta \neq 0$.

System of equations is $a^2x - by = a^2 - b$ and $bx - b^2y = 2 + 4b$. If system has unique solution, then $\Delta \neq 0$

$$\therefore \quad \begin{vmatrix} a^2 & -b \\ b & -b^2 \end{vmatrix} \neq 0$$

$$\Rightarrow \quad -a^2b^2 + b^2 \neq 0 \Rightarrow b^2(1 - a^2) \neq 0$$

$$\Rightarrow \quad b \neq 0 \text{ and } a \neq \pm 1$$

System of Three Linear Equations in Three Variables

Let the given system of linear equations in three variables x, y and z be

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$,

$$\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \ \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Then, the solution is given as

$$x = \frac{\Delta_1}{\Delta}, \ y = \frac{\Delta_2}{\Delta}, \ z = \frac{\Delta_3}{\Delta}, \text{ provided } \Delta \neq 0$$

Example 3 The solution of given system of linear equation $x + y + z = 6; \ x - y + z = 2; \ 3x + 2y - 4z = -5$, is

(a) $x = 1, y = 2, z = 3$ (b) $x = 2, y = 1, z = 3$
(c) $x = 3, y = 1, z = 2$ (d) $x = 2, y = 3, z = 1$

Sol. ***(a)*** Here, $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 3 & 2 & -4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & -2 & 0 \\ 3 & -1 & -7 \end{vmatrix} = 14$

$$\Delta_1 = \begin{vmatrix} 6 & 1 & 1 \\ 2 & -1 & 1 \\ -5 & 2 & -4 \end{vmatrix} = \begin{vmatrix} 6 & 1 & 1 \\ -4 & -2 & 0 \\ 19 & 6 & 0 \end{vmatrix} = 14$$

$$\Delta_2 = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 2 & 1 \\ 3 & -5 & -4 \end{vmatrix} = \begin{vmatrix} 1 & 6 & 1 \\ 0 & -4 & 0 \\ 0 & -23 & -7 \end{vmatrix} = 28$$

and $\Delta_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & -1 & 2 \\ 3 & 2 & -5 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 6 \\ 0 & -2 & -4 \\ 0 & -1 & -23 \end{vmatrix} = 42$

Hence, by Cramer's rule

$$x = \frac{\Delta_1}{\Delta} = 1, \ y = \frac{\Delta_2}{\Delta} = 2, \ z = \frac{\Delta_3}{\Delta} = 3$$

Nature of Solution

(i) If $\Delta \neq 0$, then the given system of equation is consistent and it has unique solution.

(ii) If $\Delta = 0$ and any of $\Delta_1, \Delta_2, \Delta_3$ is non-zero, then given system of equation is inconsistent and it will have no solution.

(iii) If all of $\Delta, \Delta_1, \Delta_2$ and Δ_3 are zero, then the given system of equation is consistent and has infinitely many solutions.

Example 4 If the system of equations

$2x + py + 6z = 8; \ x + 2y + qz = 5; \ x + y + 3z = 4$

has unique solution, then

(a) $p \neq 3, q \neq 2$ (b) $p = 3, q = 2$
(c) $p \neq 2, q \neq 3$ (d) $p = 2, q = 3$

Sol. ***(c)*** The given system of equations is

$$2x + py + 6z = 8, \quad x + 2y + qz = 5, \quad x + y + 3z = 4$$

$$\therefore \quad \Delta = \begin{vmatrix} 2 & p & 6 \\ 1 & 2 & q \\ 1 & 1 & 3 \end{vmatrix} = (2 - p)(3 - q)$$

For unique solution, $\Delta \neq 0$

$\therefore \quad (2 - p)(3 - q) \neq 0$

$\Rightarrow \quad p \neq 2$ and $q \neq 3$

Method II

This method is known as matrix method.

Let $AX = B$ be a system of linear equation. (either two or three variables). If A is non-singular, i.e. $|A| \neq 0$, then A^{-1} exists.

$$\therefore \quad AX = B$$
$$\Rightarrow \quad A^{-1}(AX) = A^{-1}B$$
$$\Rightarrow \quad (A^{-1}A)X = A^{-1}B$$
$$\Rightarrow \quad IX = A^{-1}B$$
$$\Rightarrow \quad X = A^{-1}B$$

Example 5 The solution of the following system of equations $x + 2y + z = 7; \ x + 3z = 11; \ 2x - 3y = 1$ is given by

(a) $x = 1, y = 2, z = 3$ (b) $x = 2, y = 1, z = 3$
(c) $x = 3, y = 2, z = 3$ (d) $x = 1, y = 3, z = 2$

Sol. ***(b)*** The given system of equations is

$x + 2y + z = 7; \ x + 0y + 3z = 11; \ 2x - 3y + 0z = 1$

or $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix}$

or $AX = B$, where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}, \ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix}$$

Now, $|A| = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{vmatrix} = 18$

So, the given system of equations has a unique solution given by $X = A^{-1}B$. Therefore,

$$A^{-1} = \frac{1}{|A|} \text{adj}(A) = \frac{1}{18} \begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}$$

$$\therefore \quad X = A^{-1}B = \frac{1}{18} \begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix} \begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix}$$

$$= \frac{1}{18} \begin{bmatrix} 36 \\ 18 \\ 54 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$$\Rightarrow \quad x = 2, \ y = 1, \ z = 3$$

SPEED BOOSTER

If A is a 2×2 non-singular matrix such that

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ then its inverse is}$$

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Nature of Solution

(i) If $|A| \neq 0$, then the system is consistent and has the unique solution.

(ii) If $|A| = 0$ and (adj A) $B = 0$, then the system is consistent and has infinitely many solutions.

(iii) If $|A| = 0$ and (adj A) $B \neq 0$, then the system is inconsistent and has no solution.

Example 6 The following system of equations $2x + 4y + 6z = 8$, $x + 2y + 3z = 5$, $x + y + 3z = 4$ is

(a) consistent with unique solution
(b) consistent with infinite solution
(c) inconsistent
(d) None of the above

Sol. ***(c)*** **PROBLEM SOLVING STRATEGY** Firstly, rewrite the given equation in the form of $AX = B$. Then, find adj(A) and $|A|$. If $|A| = 0$ and (adj A) $(B) \neq 0$, then system is inconsistent.

The given system is $\begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 1 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \\ 4 \end{bmatrix}$

or $AX = B$

where, $A = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 1 & 1 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 8 \\ 5 \\ 4 \end{bmatrix}$

Now, adj $(A) = \begin{bmatrix} 3 & 0 & -1 \\ -6 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ and $|A| = 0$

Also, (adj A) $(B) = \begin{bmatrix} 3 & 0 & -1 \\ -6 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 8 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 20 \\ -40 \\ 0 \end{bmatrix}$

Thus, the system of equations is inconsistent.

Method III

This method is known as Rank Method.

The rank of a matrix is the order of the highest order non-singular square matrix. Consider the system of linear equations $AX = B$.

The matrix A is called the coefficient matrix and the matrix $[A : B]$ is called augmented matrix.

Reduce the augmented matrix to echelon form by applying a sequence of elementary row operations.

Determine the number of non-zero rows in A and $[A : B]$ to determine the ranks of A and $[A : B]$. Now,

(i) If $r(A) = r(A : B)$ = number of variables, then system has unique solution.

(ii) If $r(A) = r(A : B) <$ number of variables, then system has infinite solution.

(iii) If $r(A) \neq r(A : B)$, then the system has no solution.

Example 7 For what value of λ, the system of equations $x + y + z = 6$; $x + 2y + 3z = 10$; $x + 2y + \lambda z = 12$ is inconsistent?

(b) $\lambda = 1$ *(b)* $\lambda = 2$ *(c)* $\lambda = -2$ *(d)* $\lambda = 3$

Sol. ***(d)*** The given system of equations may be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ 12 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & \lambda - 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 6 \end{bmatrix}$$

[applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$]

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & \lambda - 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$ [applying $R_3 \to R_3 - R_2$]

For $\lambda = 3$, we observe that rank of matrix A is 2 and that of the augmented matrix is 3. So, the system is inconsistent.

Solution of Homogeneous System of Simultaneous Linear Equation

Let the three homogeneous linear equations in three unknown x, y, z be

$$a_1x + b_1y + c_1z = 0, \qquad a_2x + b_2y + c_2z = 0$$
$$a_3x + b_3y + c_3z = 0$$

Clearly, $x = 0, y = 0, z = 0$ is a solution of above system of equation. This solution is known as trivial solution. Any other solution is called a non-trivial solution.

Nature of Solution

(i) If $\Delta \neq 0$ or $|A| \neq 0$, then given system of equations has only trivial solution.

(ii) If $\Delta = 0$ or $|A| = 0$, then given system of equations has non-trivial solution as well as trivial solution and the number of solutions in this case is infinite.

Example 8 If $2ax - 2y + 3z = 0$, $x + ay + 2z = 0$ and $2x + az = 0$ have a non-trivial solution, then the value of a is

(a) –1 *(b)* 0 *(c)* 1 *(d)* 2

Sol. ***(d)*** For non-trivial solution, we must have $\begin{vmatrix} 2a & -2 & 3 \\ 1 & a & 2 \\ 2 & 0 & a \end{vmatrix} = 0$

$$\Rightarrow \quad 2a(a^2 - 0) + 2(a - 4) + 3(0 - 2a) = 0$$
$$\Rightarrow \quad 2a^3 + 2a - 8 + 0 - 6a = 0$$
$$\Rightarrow \quad 2a^3 - 4a - 8 = 0$$
$$\Rightarrow \quad a^3 - 2a - 4 = 0$$
$$\Rightarrow \quad (a - 2)(a^2 + 2a + 2) = 0$$
$$\Rightarrow \quad a = 2$$

Example 9 Let a, b, c be any real numbers, suppose that there are real numbers x, y, z not all zero such that $x = cy + bz$, $y = az + cx$ and $z = bx + ay$. Then, $a^2 + b^2 + c^2 + 2abc$ is equal to

(a) 0 (b) 1 (c) 2 (d) 3

Sol. **(b)** It is given that there are x, y, z not all zero satisfying $x = cy + bz$, $y = az + cx$ and $z = bx + ay$. So, the system of equations $x - cy - bz = 0$, $cx - y + az = 0$, $bx + ay - z = 0$ has non-trivial solutions.

$$\therefore \quad \begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 0$$

$$\Rightarrow \quad 1 - a^2 - c^2 - b^2 - 2abc = 0$$

$$\Rightarrow \quad a^2 + b^2 + c^2 + 2abc = 1$$

Rank Method

In case of homogeneous system of linear equations the rank of the augmented matrix is always same as that of the coefficient matrix. So, a homogeneous system of linear equation is always consistent.

If $r(A)$ = number of variables, then $AX = 0$ has a unique solution $X = 0$, i.e. $x_1 = x_2 = x_3 = \ldots = x_n = 0$.

If $r(A) <$ number of variables, then the system of equations has infinitely many solutions.

Smart Practice

1. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 5 \\ 0 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$, then which of the following is true?

(a) $AX = B$ has a unique solution
(b) $AX = B$ has exactly three solutions
(c) $AX = B$ has infinitely many solutions
(d) $AX = B$ is inconsistent.

2. The number of 3×3 matrices A whose entries are either 0 or 1 and for which the system $AX = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has exactly two distinct solutions, is

(a) 0 (b) $2^9 - 1$ (c) 168 (d) 2

3. If a system of three linear equations $x + 4ay + a = 0$, $x + 3by + b = 0$ and $x + 2cy + c = 0$ is consistent, then a, b, c are in

(a) AP (b) GP (c) HP (d) None

4. The value of λ for which the system of equations $x + y - 2z = 0$, $2x - 3y + z = 0$, $x - 5y + 4z = \lambda$ is consistent, is

(a) 0 (b) 1 (c) 2 (d) 3

5. Consider the following system of linear equations

$x + ky + 3z = 0$, $3x + ky - 2z = 0$, $2x + 3y - 4z = 0$

Then, the value of k for which system possesses a non-trivial solution, is

(a) 11/2 (b) 11 (c) 33/2 (d) 22

6. a, b, c are distinct real numbers, not equal to one. If $ax + y + z = 0$, $x + by + z = 0$ and $x + y + cz = 0$ have a non-trivial solution, then the value of $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c}$ is

(a) -1 (b) 1 (c) 0 (d) None of these

7. If α, β, γ are the angles of a triangle and the system of equations

$\cos(\alpha - \beta)x + \cos(\beta - \gamma)y + \cos(\gamma - \alpha)z = 0$

$\cos(\alpha + \beta)x + \cos(\beta + \gamma)y + \cos(\gamma + \alpha)z = 0$

$\sin(\alpha + \beta)x + \sin(\beta + \gamma)y + \sin(\gamma + \alpha)z = 0$

has non-trivial solutions, then triangle is necessarily

(a) equilateral (b) right angled
(c) isosceles (d) acute angled

8. Given $a = x/(y - z)$, $b = y/(z - x)$ and $c = z/(x - y)$, where x, y and z are not all zero, then the value of $ab + bc + ca$ is

(a) 0 (b) 1 (c) -1 (d) None of these

9. If a, b, c are in GP with common ratio r_1 and α, β, γ are in GP, with common ratio r_2 and equations $ax + \alpha y + z = 0$, $bx + \beta y + z = 0$, $cx + \gamma y + z = 0$ have only zero solution, then which of the following is not true?

(a) $r_1 \neq 1$ (b) $r_2 \neq 1$
(c) $r_1 \neq r_2$ (d) None of these

10. If $a > b > c$ and the system of equations $ax + by + cz = 0$, $bx + cy + az = 0$ and $cx + ay + bz = 0$ has a non-trivial solution, then the quadratic equation $ax^2 + bx + c = 0$ has

(a) atleast one positive root (b) roots opposite in sign
(c) positive roots (d) imaginary roots

11. If $x^a y^b = e^m$, $x^c y^d = e^n$, $\Delta_1 = \begin{vmatrix} m & b \\ n & d \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} a & m \\ c & n \end{vmatrix}$ and $\Delta_3 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$, then the values of x and y are

(a) $\frac{\Delta_1}{\Delta_3}$ and $\frac{\Delta_2}{\Delta_3}$ (b) $\frac{\Delta_2}{\Delta_1}$ and $\frac{\Delta_3}{\Delta_1}$

(c) $\log\left(\frac{\Delta_1}{\Delta_3}\right)$, $\log\left(\frac{\Delta_2}{\Delta_3}\right)$ (d) e^{Δ_1/Δ_3} and e^{Δ_2/Δ_3}

12. The existence of the unique solution of the system of equations
$x + y + z = \lambda,\ 5x - y + \mu z = 10,\ 2x + 3y - z = 6$
depends on
(a) only μ *(b)* only λ
(c) Both λ and μ *(d)* neither λ nor μ

13. **Statement 1** If $bc + qr = ca + rp = ab + pq = -1$, then

$$\begin{vmatrix} ap & a & p \\ bq & b & q \\ cr & c & r \end{vmatrix} = 0 \ (abc, pqr \neq 0)$$

Statement 2 If system of equations
$a_1x + b_1y + c_1 = 0,\ a_2x + b_2y + c_2 = 0,$
$a_3x + b_3y + c_3 = 0$ has non-trivial solutions, then

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

(a) Both are true *(b)* Only statement 1 is true
(c) Only statement 2 is true *(d)* Both are false

14. The values of $k \in R$ for which the system of equations $x + ky + 3z = 0,\ kx + 2y + 2z = 0,$ $2x + 3y + 4z = 0$ admits of non-trivial solution, is/are
(a) 2, 5/4 *(b)* 5/2, 2 *(c)* 3, 2 *(d)* 5/4, 3

15. The system of equations
$-2x + y + z = a;\ x - 2y + z = b;\ x + y - 2z = c$ has
(a) no solution, if $a + b + c = 0$
(b) unique solution, if $a + b + c = 0$
(c) infinite number of solutions, if $a + b + c = 0$
(d) None of the above

16. The sum of values of p for which the equations $x + y + z = 1,\ x + 2y + 4z = p$ and $x + 4y + 10z = p^2$ have a solution is
(a) 1 *(b)* 3 *(c)* 5 *(d)* 7

17. The value of $|\alpha|$ for which the system of equation
$\alpha x + y + z = \alpha - 1, x + \alpha y + z = \alpha - 1, x + y + \alpha z = \alpha - 1$
has no solution, is
(a) 2 *(b)* 1 *(c)* 0 *(d)* None

18. The equation $\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 4 \\ 3 & 4 & k \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has a solution for (x, y, z) besides $(0, 0, 0)$. Then, the value of k is
(a) -1 *(b)* 0 *(c)* 1 *(d)* 2

19. Let α, β, γ be the real roots of the equation $x^3 + ax^2 + bx + c = 0$ $(a, b, c \in R$ and $a \neq 0)$. If the system of equations (in u, v and w) given by
$\alpha u + \beta v + \gamma w = 0, \beta u + \gamma v + \alpha w = 0$
and $\gamma u + \alpha v + \beta w = 0$
has non-trivial solutions, then the value of a^2/b is
(a) $a^2 + 3b = 0$ *(b)* $a^2 - 3b = 0$
(c) $a^2 - 2b = 0$ *(d)* $a^2 + 2b = 0$

20. Consider the system of equations
$x + y + z = 6,\ x + 2y + 3z = 10,\ x + 2y + \lambda z = \mu$
The system has unique solution, if
(a) $\lambda \neq 3$ *(b)* $\lambda = 3, \mu = 10$
(c) $\lambda = 3, \mu \neq 0$ *(d)* None of these

21. Given that the system of equations $x = cy + bz$, $y = az + cx$, $z = bx + ay$ has non-zero solutions and atleast one of the a, b, c is a proper fraction, then $a^2 + b^2 + c^2$ is
(a) > 2 *(b)* > 3 *(c)* < 3 *(d)* < 2

22. If the system of linear equations
$x + 2ay + az = 0,\ x + 3by + az = 0,\ x + 4cy + cz = 0$
has a non-zero solution, then a, b, c
(a) satisfy $a + 2b + 3c = 0$ *(b)* are in AP
(c) are in GP *(d)* are in HP

23. Consider the system of equations in x, y, z as
$x \sin 3\theta - y + z = 0,\quad x \cos 2\theta + 4y + 3z = 0$
$2x + 7y + 7z = 0$
If this system has a non-trivial solution, then for any integer n, values of θ are given
(a) $\left[n + \frac{(-1)^n}{3}\right]\pi$ *(b)* $\left[n + \frac{(-1)^n}{4}\right]\pi$ *(c)* $\left[n + \frac{(-1)^n}{6}\right]\pi$ *(d)* $\frac{n\pi}{2}$

24. Consider the equations
$ax + by + cz = d,\quad a^2x + b^2y + c^2z = d^2$
$a^3x + b^3y + c^3z = d^3$
Match the Column I with the values of x, y, z given in Column II.

Column I	Column I
(A) x	(P) $\frac{d(a-d)(d-c)}{b(a-b)(b-c)}$
(B) y	(Q) $\frac{d(b-d)(d-a)}{c(b-c)(c-a)}$
(C) z	(R) $\frac{d(d-b)(c-d)}{a(a-b)(c-a)}$

Codes

	A	B	C
(a)	R	P	Q
(c)	Q	P	R

	A	B	C
(b)	P	Q	R
(d)	Q	R	P

25. Consider the system of equations
$(a-1)x - y - z = 0$
$x - (b-1)y + z = 0$
$x + y - (c-1)z = 0$
where, a, b and c are non-zero real numbers.
Statement I If x, y, z are not all zero, then $ab + bc + ca = abc$.
Statement II $abc \geq 27$
(a) Only Statement I is correct *(b)* Only Statement II is correct
(c) Both are correct *(d)* Both are incorrect

Answers with Explanation

1. ***(a)*** $|A| = 1(0-10) - 2(2-6) + 3(4-0)$

$= -10 + 8 + 12 = 10$

$\therefore \quad |A| \neq 0$

Hence, it has a unique solution.

2. ***(a)*** Three planes cannot intersect at two distinct points.

3. ***(c)*** Since, system of equations is consistent.

So, we have $\Delta = \begin{vmatrix} 1 & 4a & a \\ 1 & 3b & b \\ 1 & 2c & c \end{vmatrix} = 0$

$\Rightarrow \quad 1(3bc - 2bc) - 4a(c-b) + a(2c-3b) = 0$

$\Rightarrow \quad bc - 4ac + 4ab + 2ac - 3ab = 0$

$\Rightarrow \quad bc - 2ac + ab = 0$

$\Rightarrow \quad bc + ab = 2ac$

$\Rightarrow \quad \frac{1}{a} + \frac{1}{c} = \frac{2}{b}$

Hence, a, b, c are in HP.

4. ***(a)*** The given system is

$x - 5y + 4z = \lambda,\ x + y - 2z = 0,\ 2x - 3y + z = 0$

$\therefore \quad \Delta = \begin{vmatrix} 1 & -5 & 4 \\ 1 & 1 & -2 \\ 2 & -3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -5 & 4 \\ 0 & 6 & -6 \\ 0 & 7 & -7 \end{vmatrix} = 0$

Hence, system is consistent only when $\Delta_1 = \Delta_2 = \Delta_3 = 0$

Now, $\Delta_1 = \begin{vmatrix} \lambda & -5 & 4 \\ 0 & 1 & -2 \\ 0 & -3 & 1 \end{vmatrix} = -5\lambda = 0 \Rightarrow \lambda = 0$

For $\lambda = 0$, clearly, $\Delta_2 = \Delta_3 = 0$

5. ***(c)*** The system $x + ky + 3z = 0,\ 3x + ky - 2z = 0, 2x + 3y - 4z = 0$ has non-trivial solution, if $\Delta = 0$

$\therefore \quad \Delta = \begin{vmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 3 & -4 \end{vmatrix} = 0$

$\Rightarrow \quad 1(-4k+6) - k(-12+4) + 3(9-2k) = 0$

$\Rightarrow \quad -4k + 6 + 8k + 27 - 6k = 0 \Rightarrow -2k + 33 = 0$

$\Rightarrow \quad k = \frac{33}{2}$

6. ***(b)*** Since, the system has non-trivial solution, we have

$$\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} a-1 & 1-b & 0 \\ 0 & b-1 & 1-c \\ 1 & 1 & c \end{vmatrix} = 0$$

$\Rightarrow \quad c(1-a)(1-b) + (1-b)(1-c) - (1-c)(a-1) = 0$

Dividing throughout by $(1-a)(1-b)(1-c)$, we get

$$\frac{c}{1-c} + \frac{1}{1-a} + \frac{1}{1-b} = 0 \Rightarrow -1 + \frac{1}{1-c} + \frac{1}{1-a} + \frac{1}{1-b} = 0$$

$\Rightarrow \quad \frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$

7. ***(c)*** For non-trivial solution, $\Delta = 0$

$$\therefore \begin{vmatrix} \cos(\alpha-\beta) & \cos(\beta-\gamma) & \cos(\gamma-\alpha) \\ \cos(\alpha+\beta) & \cos(\beta+\gamma) & \cos(\gamma+\alpha) \\ \sin(\alpha+\beta) & \sin(\beta+\gamma) & \sin(\gamma+\alpha) \end{vmatrix} = 0$$

$\Rightarrow \quad$ Either $\alpha = \beta$ or $\beta = \gamma$ or $\gamma = \alpha$.

$\therefore$ Triangle is isosceles.

8. ***(c)*** We have,

$a = x/(y-z) \Rightarrow x - ay + az = 0$

$b = y/(z-x) \Rightarrow bx + y - bz = 0$

$c = z/(x-y) \Rightarrow -cx + cy + z = 0$

Since, x, y, z are not all zero, then the above system has a non-trivial solution. So, $\Delta = 0$

$$\Rightarrow \quad \begin{vmatrix} 1 & -a & a \\ b & 1 & -b \\ -c & c & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad 1(1+bc) + a(b-bc) + a(bc+c) = 0$

$\Rightarrow \quad 1 + bc + ab - abc + abc + ca = 0$

$\Rightarrow \quad 1 + ab + bc + ca = 0$

$\Rightarrow \quad ab + bc + ca = -1$

9. ***(d)*** As a, b, c are in GP with common ratio r_1 and α, β, γ are in GP having common ratio r_2, $a \neq 0, \alpha \neq 0$, $b = ar_1, c = ar_1^2$, $\beta = ar_2$, $\gamma = ar_2^2$.

Also, the system of equations has only zero (trivial) solution.

$$\therefore \quad \Delta = \begin{vmatrix} a & \alpha & 1 \\ b & \beta & 1 \\ c & \gamma & 1 \end{vmatrix} \neq 0$$

$$\Rightarrow \quad \begin{vmatrix} a & \alpha & 1 \\ ar_1 & \alpha r_2 & 1 \\ ar_1^2 & \alpha r_2^2 & 1 \end{vmatrix} \neq 0 \Rightarrow a\alpha \begin{vmatrix} 1 & 1 & 1 \\ r_1 & r_2 & 1 \\ r_1^2 & r_2^2 & 1 \end{vmatrix} \neq 0$$

$\Rightarrow \quad a\alpha(r_1-1)(r_2-1)(r_1-r_2) \neq 0 \Rightarrow r_1 \neq 1,\ r_2 \neq 1,\ r_1 \neq r_2$

10. ***(a)*** The given system of equations has a non-trivial solution.

$\therefore \quad \Delta = 0$

$$\Rightarrow \quad \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

$\Rightarrow \quad (a+b+c)(a^2+b^2+c^2-ab-bc-ca) = 0$

$\Rightarrow \quad (a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2] = 0$

$\Rightarrow \quad a+b+c = 0$

$\Rightarrow\ x = 1$ is a root of $ax^2 + bx + c = 0$.

Thus, the quadratic equation has atleast one positive root.

11. ***(d)*** We have,

$$x^a y^b = e^m$$

$\Rightarrow \quad a\log x + b\log y = m$

and $\quad x^c y^d = e^n$

$\Rightarrow \quad c\log x + d\log y = n$

Using Cramer's rule, we have

$$\log x = \frac{\begin{vmatrix} m & b \\ n & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \text{ and } \log y = \frac{\begin{vmatrix} a & m \\ c & n \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

$\Rightarrow \quad \log x = \frac{\Delta_1}{\Delta_3}$ and $\log y = \frac{\Delta_2}{\Delta_3}$

$\Rightarrow \quad x = e^{\Delta_1/\Delta_3}$ and $y = e^{\Delta_2/\Delta_3}$

12. ***(a)*** The given system of equations may be written as

$$\begin{bmatrix}1 & 1 & 1\\ 5 & -1 & \mu\\ 2 & 3 & -1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}\lambda\\ 10\\ 6\end{bmatrix}\Rightarrow\begin{bmatrix}1 & 1 & 1\\ 0 & -6 & \mu-5\\ 0 & 1 & -3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}\lambda\\ 10-5\lambda\\ 6-2\lambda\end{bmatrix}$$

$$\Rightarrow\quad\begin{bmatrix}1 & 1 & 1\\ 0 & -6 & \mu-5\\ 0 & 6 & -18\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}\lambda\\ 10-5\lambda\\ 36-12\lambda\end{bmatrix}$$

$$\Rightarrow\quad\begin{bmatrix}1 & 1 & 1\\ 0 & -6 & \mu-5\\ 0 & 0 & \mu-23\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}\lambda\\ 10-5\lambda\\ 46-17\lambda\end{bmatrix}$$

For unique solution we must have

$$\mu-23\neq 0$$

i.e. $\mu\neq 23$

Hence, the existence of the unique solution depends on μ only.

13. ***(a)*** We have,

$$1+bc+qr=0 \quad\ldots(i)$$
$$1+ca+pr=0 \quad\ldots(ii)$$
$$1+ab+pq=0 \quad\ldots(iii)$$

The determinant in the question involves a column consisting the element ap, bq and cr. So, multiplying Eqs. (i), (ii) and (iii) by ap, bq and cr, respectively, we get

$$ap+abcp+apqr=0 \quad\ldots(iv)$$
$$bq+abcq+bpqr=0 \quad\ldots(v)$$
$$cq+abcr+cpqr=0 \quad\ldots(vi)$$

Since, abc and pqr occur in all the three equations, putting $abc=x$, $pqr=y$, we get the system

$$\begin{aligned}ap+px+ay&=0\\ bq+qx+by&=0\\ cr+rx+cy&=0\end{aligned}\quad\ldots(vii)$$

System (vii) must have a common solution, i.e. system is consistent.

So, $\begin{vmatrix}ap & p & a\\ bq & q & b\\ cr & r & c\end{vmatrix}=0$ or $\begin{vmatrix}ap & a & p\\ bq & b & q\\ cr & c & r\end{vmatrix}=0$

14. ***(a)*** For non-trivial solution, $\begin{vmatrix}1 & k & 3\\ k & 2 & 2\\ 2 & 3 & 4\end{vmatrix}=0$

$$\Rightarrow\quad 1(8-6)-k(4k-4)+3(3k-4)=0$$
$$\Rightarrow\quad 2-4k^2+4k+9k-12=0$$
$$\Rightarrow\quad 4k^2-13k+10=0$$
$$\Rightarrow\quad 4k^2-8k-5k+10=0$$
$$\Rightarrow\quad 4k(k-2)-5(k-2)=0$$
$$\Rightarrow\quad (k-2)(4k-5)=0$$
$$\Rightarrow\quad k=2,5/4$$

15. ***(c)*** We have, $\Delta=\begin{vmatrix}-2 & 1 & 1\\ 1 & -2 & 1\\ 1 & 1 & -2\end{vmatrix}=0$

$$\Delta_1=\begin{vmatrix}a & 1 & 1\\ b & -2 & 1\\ c & 1 & -2\end{vmatrix}=3a+3b+3c$$

$$\Delta_2=\begin{vmatrix}-2 & a & 1\\ 1 & b & 1\\ 1 & c & -2\end{vmatrix}=3a+3b+3c$$

$$\Delta_3=\begin{vmatrix}-2 & 1 & a\\ 1 & -2 & b\\ 1 & 1 & c\end{vmatrix}=3a+3b+3c$$

For no solution, $a+b+c\neq 0$ and for infinite number of solution $a+b+c=0$.

16. ***(b)*** $\because\ x+y+z=1,\ x+2y+4z=p,\ x+4y+10z=p^2$

$$\Delta=\begin{vmatrix}1 & 1 & 1\\ 1 & 2 & 4\\ 1 & 4 & 10\end{vmatrix}=1(20-16)-1(10-4)+1(4-2)=4-6+2=0$$

Since $\Delta=0$, solution is not unique solution. The system will have infinite solutions, if $\Delta_1=0$, $\Delta_2=0$, $\Delta_3=0$.

$$\Delta_1=\begin{vmatrix}1 & 1 & 1\\ p & 2 & 4\\ p^2 & 4 & 10\end{vmatrix}=0$$

$$\Rightarrow\quad 1(20-16)-1(10p-4p^2)+1(4p-2p^2)=0$$
$$\Rightarrow\quad 4-10p+4p^2+4p-2p^2=0$$
$$\Rightarrow\quad 2p^2-6p+4=0$$
$$\Rightarrow\quad p^2-3p+2=0$$
$$\Rightarrow\quad (p-1)(p-2)=0$$
$$\Rightarrow\quad p=1,2$$

Also, for these values of p, Δ_2, $\Delta_3=0$.

17. ***(a)*** System of equations $ax+y+z=\alpha-1$, $x+\alpha y+z=\alpha-1$, $x+y+\alpha z=\alpha-1$. Since, system has no solution.

Therefore, $\Delta=0$ and $\alpha-1\neq 0$

$$\therefore\quad\begin{vmatrix}\alpha & 1 & 1\\ 1 & \alpha & 1\\ 1 & 1 & \alpha\end{vmatrix}=0,\ \alpha\neq 1$$

$$\Rightarrow\quad \alpha(\alpha^2-1)-1(\alpha-1)+1(1-\alpha)=0,\alpha\neq 1$$
$$\Rightarrow\quad (\alpha-1)[\alpha(\alpha+1)-1-1]=0,\ \alpha\neq 1$$
$$\Rightarrow\quad (\alpha-1)(\alpha^2+\alpha-2)=0,\ \alpha\neq 1$$
$$\Rightarrow\quad (\alpha-1)(\alpha-1)(\alpha+2)=0,\alpha\neq 1$$
$$\Rightarrow\quad \alpha=1,-2\ ,\alpha\neq 1$$
$$\therefore\quad \alpha=-2 \text{ and } |\alpha|=2$$

18. ***(d)*** $\begin{vmatrix}1 & 2 & 2\\ 1 & 3 & 4\\ 3 & 4 & k\end{vmatrix}=0\Rightarrow 1(3k-16)-2(k-12)+2(4-9)=0$

$\Rightarrow\ 3k-16-2k+24-10=0\ \Rightarrow\ k=2$

19. ***(b)*** Equation $x^3+ax^2+bx+c=0$ has roots α, β, γ. Therefore, $\alpha+\beta+\gamma=-a$, $\alpha\beta+\beta\gamma+\gamma\alpha=b$

Since, the given system of equations has non-trivial solution, we have

$$\begin{vmatrix}\alpha & \beta & \gamma\\ \beta & \gamma & \alpha\\ \gamma & \alpha & \beta\end{vmatrix}=0$$

$$\Rightarrow\quad \alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma=0$$
$$\Rightarrow\quad (\alpha+\beta+\gamma)[\alpha^2+\beta^2+\gamma^2-\alpha\beta-\beta\gamma-\gamma\alpha]=0$$

$\Rightarrow (\alpha+\beta+\gamma)[(\alpha+\beta+\gamma)^2-3(\alpha\beta+\beta\gamma+\gamma\alpha)]=0$

$\Rightarrow -a(a^2-3b)=0 \Rightarrow a^2/b=3$

$\Rightarrow a^2-3b=0$

20. ***(a)*** $\Delta=\begin{vmatrix}1&1&1\\1&2&3\\1&2&\lambda\end{vmatrix}=2\lambda+3+2-2-\lambda-6=\lambda-3$

$\Delta_1=\begin{vmatrix}6&1&1\\10&2&3\\\mu&2&\lambda\end{vmatrix}=12\lambda+3\mu+20-2\mu-10\lambda-36=2\lambda+\mu-16$

$\Delta_2=\begin{vmatrix}1&6&1\\1&10&3\\1&\mu&\lambda\end{vmatrix}=10\lambda+18+\mu-10-3\mu-6\lambda=4\lambda-2\mu+8$

$\Delta_3=\begin{vmatrix}1&1&6\\1&2&10\\1&2&\mu\end{vmatrix}=2\mu+10+12-12-\mu-20=\mu-10$

Thus, the system has unique solution, if $\Delta\neq0$ or $\lambda\neq3$.

21. ***(c)*** The system of equations

$-x+cy+bz=0$ ...(i)

$cx-y+az=0$ ...(ii)

$bx+ay-z=0$ ...(iii)

has a non-zero solution, if $\Delta=\begin{vmatrix}-1&c&b\\c&-1&a\\b&a&-1\end{vmatrix}=0$

$\Rightarrow a^2+b^2+c^2+2abc-1=0$

or $a^2+b^2+c^2+2abc=1$ ...(iv)

Then, clearly the system has infinitely many solutions. From Eqs. (i) and (ii), we have

$$\frac{x}{ac+b}=\frac{y}{bc+a}=\frac{z}{1-c^2}$$

$$\therefore \frac{x^2}{(ac+b)^2}=\frac{y^2}{(bc+a)^2}=\frac{z^2}{(1-c^2)^2}$$

$$\Rightarrow \frac{x^2}{(1-a^2)(1-c^2)}=\frac{y^2}{(1-b^2)(1-c^2)}=\frac{z^2}{(1-c^2)^2} \quad \text{[from Eq. (iv)]}$$

$$\Rightarrow \frac{x^2}{1-a^2}=\frac{y^2}{1-b^2}=\frac{z^2}{1-c^2} \quad \text{...(v)}$$

From Eq. (v), we see that $1-a^2, 1-b^2, 1-c^2$ are all positive or all negative. Given that one of a, b, c is proper fraction, so $1-a^2>0, 1-b^2>0, 1-c^2>0$, which gives $a^2+b^2+c^2<3$.

22. ***(d)*** Given system of equations will have a non-zero solution, if

$$\begin{vmatrix}1&2a&a\\1&3b&b\\1&4c&c\end{vmatrix}=0 \Rightarrow \begin{vmatrix}1&2a&a\\0&3a-2b&b-a\\0&4c-2a&c-a\end{vmatrix}=0$$

[applying $R_2\to R_2-R_1, R_3\to R_3-R_1$]

$\Rightarrow (3b-2a)(c-a)(b-a)(4c-2a)=0$

[expanding along R_1]

$\Rightarrow (3bc-3ab-2ca+2a^2)-(4bc-2ab-4ca+2a^2)=0$

$\Rightarrow -bc-ab+2ca=0 \Rightarrow \frac{2}{b}=\frac{1}{a}+\frac{1}{c}$

$\therefore$ a, b, c are in HP.

23. ***(c)*** Given system of equations has a non-trivial solution.

$$\therefore \begin{vmatrix}\sin3\theta&-1&1\\\cos2\theta&4&3\\2&7&7\end{vmatrix}=0$$

$\Rightarrow 7\sin3\theta+14\cos2\theta-14=0$

$\Rightarrow 3\sin\theta-4\sin^3\theta+2-4\sin^2\theta-2=0$

$\Rightarrow \sin\theta(3-4\sin^2\theta-4\sin\theta)=0$

$\Rightarrow \sin\theta(2\sin\theta+3)(2\sin\theta-1)=0$

$\Rightarrow \sin\theta=0$ or $\sin\theta=\frac{1}{2}$

$\Rightarrow \theta=n\pi$ or $\theta=n\pi+(-1)^n\frac{\pi}{6}$

24. ***(a)*** Here, we use Cramer's rule to solve this system.

$$\therefore \Delta=\begin{vmatrix}a&b&c\\a^2&b^2&c^2\\a^3&b^3&c^3\end{vmatrix}=abc\begin{vmatrix}1&1&1\\a&b&c\\a^2&b^2&c^2\end{vmatrix}$$

$=abc(a-b)(b-c)(c-a)$

Since, $\Delta\neq0$ i.e. a unique solution, therefore, if a, b, c are distinct and non-zero, then the system has a unique solution.

Now, $\Delta_x=\begin{vmatrix}d&b&c\\d^2&b^2&c^2\\d^3&b^3&c^3\end{vmatrix}=dbc(d-b)(b-c)(c-d)$

$\Delta_y=\begin{vmatrix}a&d&c\\a^2&d^2&c^2\\a^3&d^3&c^3\end{vmatrix}=adc(a-d)(d-c)(c-a)$

and $\Delta_z=\begin{vmatrix}a&b&d\\a^2&b^2&d^2\\a^3&b^3&d^3\end{vmatrix}=abd(a-b)(b-d)(d-a)$

$$\therefore x=\frac{\Delta_x}{\Delta}=\frac{d(d-b)(c-d)}{a(a-b)(c-a)}$$

$$y=\frac{\Delta_y}{\Delta}=\frac{d(a-d)(d-c)}{b(a-b)(b-c)}$$

and $$z=\frac{\Delta_z}{\Delta}=\frac{d(b-d)(d-a)}{c(b-c)(c-a)}$$

25. ***(c)*** I. It is given that x, y, z are not all zero.

So, the given system of equations has a non-trivial solutions.

$$\therefore \begin{vmatrix}a-1&-1&-1\\1&-(b-1)&1\\1&1&-(c-1)\end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix}a&0&-1\\0&-b&1\\c&c&1-c\end{vmatrix}=0$$

[applying $C_1\to C_1-C_3, C_2\to C_2-C_3$]

$\Rightarrow a(-b+bc-c)-(0+bc)=0$

$\Rightarrow ab+bc+ac=abc$

II. Using AM $\geq$ GM, we have

$$\frac{ab+bc+ac}{3}\geq(abc)^{2/3} \Rightarrow abc\geq3(abc)^{2/3}$$

$\Rightarrow (abc)^{1/3}\geq3$

$\Rightarrow abc\geq27$

So, both statements are correct.

RELATION AND FUNCTION, INVERSE TRIGONOMETRIC FUNCTION

FOR CLASS XII

Topicwise Collection of best subjective problems

Very Short Answer Type Questions [1 Mark]

1. If a relation R is defined on the set Z of integers as follows $(a, b) \in R \Leftrightarrow a^2 + b^2 = 25$. Find the domain (R).

2. If a set A has 3 elements and set B has 4 elements, then find the number of injections that can be defined from A to B.

3. If $f(x) = \frac{1-x}{1+x}$, $x \neq -1$, then find $f^{-1}(x)$ in terms of $f(x)$.

4. Let '$*$' be a binary operation of N given by $a * b = \text{LCM}(a, b)$ for all $a, b \in N$. Find the identity element in N.

5. Which is greater, $\tan 1$ or $\tan^{-1} 1$?

6. If $\tan^{-1} x + \tan^{-1} y = \frac{4\pi}{5}$, then find $\cot^{-1} x + \cot^{-1} y$.

7. Prove that

$$\tan^{-1}\frac{1-x}{1+x} - \tan^{-1}\frac{1-y}{1+y} = \tan^{-1}\frac{y-x}{1+yx}.$$

8. If $\alpha = \tan^{-1}\left(\tan\frac{5\pi}{4}\right)$ and $\beta = \tan^{-1}\left(-\tan\frac{2\pi}{3}\right)$, then prove that $4\alpha = 3\beta$.

9. Find the domain of $f(x) = \sin^{-1}(-x^2)$.

10. Prove that $\tan^2(\sec^{-1}2) + \cot^2(\text{cosec}^{-1}3) = 11$.

Short Answer Type Questions [4 Marks]

11. If $f : R \to R$ is given by

$$f(x) = \sin^2 x + \sin^2\left(x + \frac{\pi}{3}\right) + \cos x \cos\left(x + \frac{\pi}{3}\right)$$

for all $x \in R$ and $g : R \to R$ be such that $g\left(\frac{5}{4}\right) = 1$, then prove that $gof : R \to R$ is a constant function.

12. Prove that $\tan^{-1}\frac{yz}{xr} + \tan^{-1}\frac{zx}{yr} + \tan^{-1}\frac{xy}{zr} = \frac{\pi}{2}$, where $x, y, z > 0$ such that $x^2 + y^2 + z^2 = r^2$.

13. Solve the equation

$$\sin^{-1} 6x + \sin^{-1} 6\sqrt{3}x = -\frac{\pi}{2}.$$

14. If $y = \cot^{-1}(\sqrt{\cos x}) - \tan^{-1}(\sqrt{\cos x})$, then prove that $\sin y = \tan^2\frac{x}{2}$.

15. Evaluate

$$\tan^{-1}\left(\frac{3\sin 2\alpha}{5 + 3\cos 2\alpha}\right) + \tan^{-1}\left(\frac{1}{4}\tan\alpha\right).$$

16. Prove that

$$\tan\left(\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\frac{a}{b}\right) + \tan\left(\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\frac{a}{b}\right) = \frac{2b}{a}.$$

Long Answer Type Questions [6 Marks]

17. Let N be the set of all natural numbers and let R be a relation on $N \times N$, defined by $(a, b)\, R\, (c, d) \Leftrightarrow ad = bc$ for all $(a, b), (c, d) \in N \times N$.

Show that R is an equivalence relation on $N \times N$. Also, find the equivalence class $[(2, 6)]$.

18. On the set $M = A(x) = \left\{\begin{bmatrix} x & x \\ x & x \end{bmatrix} : x \in R\right\}$ of 2×2 matrices, find the identify element for the multiplication of matrices as a binary operation. Also, find the inverse of an element of M.

19. Let $x, y, z \in [-1, 1]$ be such that $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{3\pi}{2}$. Find the values of

(i) $x^{2018} + y^{2019} + z^{2020}$

(ii) $x^{2016} + y^{2018} + z^{2020} - \frac{9}{x^{2016} + y^{2018} + z^{2020}}$

20. Find the greatest and least values of $(\sin^{-1} x)^2 + (\cos^{-1} x)^2$.

Answers with Explanation

1. We have, $(a, b) \in R \Leftrightarrow a^2 + b^2 = 25$

$\Rightarrow \quad b = \pm\sqrt{25 - a^2}$

Clearly, $a = 0 \Rightarrow b = \pm 5$, $a = \pm 3 \Rightarrow b = \pm 4$, $a = \pm 4$
$\Rightarrow b = \pm 3$ and $a = \pm 5 \Rightarrow b = 0$

Hence, domain $(R) = \{a : (a, b) \in R\} = \{0, \pm 3, \pm 4, \pm 5\}$

2. The total number of injective mappings from a set A containing 3 elements to a set B containing 4 elements

$= {}^4C_3 \times 3! = 24$

3. Let $f(x) = y \Rightarrow \dfrac{1 - x}{1 + x} = y$

$\Rightarrow \quad 1 - x = xy + y$

$\Rightarrow \quad 1 - y = x(1 + y)$

$\Rightarrow \quad x = \dfrac{1 - y}{1 + y}$

$\therefore \quad f^{-1}(x) = \dfrac{1 - x}{1 + x} = f(x)$

4. Let e be the identity element.

Then, $a * e = a = e * a$ for all $a \in N$

$\Rightarrow \quad a * e = a$ for all $a \in N$

$\Rightarrow \quad \text{LCM}(a, e) = a$ for all $a \in N$

$\Rightarrow \quad e = 1$

So, 1 is the identity element in N.

5. We know that, $\tan^{-1} 1 = \dfrac{\pi}{4}$ and $1 > \dfrac{\pi}{4}$

Now, $1 > \dfrac{\pi}{4}$

$\Rightarrow \quad \tan 1 > \tan\dfrac{\pi}{4}$

$\Rightarrow \quad \tan 1 > 1$

$\Rightarrow \quad \tan 1 > 1 > \dfrac{\pi}{4}$

$\Rightarrow \quad \tan 1 > \dfrac{\pi}{4}$

$\Rightarrow \quad \tan 1 > \tan^{-1} 1$

6. We have, $\tan^{-1} x + \tan^{-1} y = \dfrac{4\pi}{5}$

$\Rightarrow \dfrac{\pi}{2} - \cot^{-1} x + \dfrac{\pi}{2} - \cot^{-1} y = \dfrac{4\pi}{5} \quad \left[\because \tan^{-1} a + \cot^{-1} a = \dfrac{\pi}{2}\right]$

$\Rightarrow \quad \pi - (\cot^{-1} x + \cot^{-1} y) = \dfrac{4\pi}{5}$

$\Rightarrow \quad \cot^{-1} x + \cot^{-1} y = \pi - \dfrac{4\pi}{5}$

$\Rightarrow \quad \cot^{-1} x + \cot^{-1} y = \dfrac{\pi}{5}$

7. $\text{LHS} = \tan^{-1}\dfrac{1 - x}{1 + x} - \tan^{-1}\dfrac{1 - y}{1 + y}$

$= (\tan^{-1} 1 - \tan^{-1} x) - (\tan^{-1} 1 - \tan^{-1} y)$

$= \tan^{-1} y - \tan^{-1} x = \tan^{-1}\left(\dfrac{y - x}{1 + yx}\right)$ **Hence proved.**

8. We have, $\alpha = \tan^{-1}\left(\tan\dfrac{5\pi}{4}\right)$

$= \tan^{-1}\tan\left(\pi + \dfrac{\pi}{4}\right) = \tan^{-1}\tan\dfrac{\pi}{4}$

$\Rightarrow \quad \alpha = \dfrac{\pi}{4}$ ...(i)

Again, $\beta = \tan^{-1}\left(-\tan\dfrac{2\pi}{3}\right)$

$= \tan^{-1}\left(-\tan\left(\pi - \dfrac{\pi}{3}\right)\right)$

$= \tan^{-1}\left(-\left(-\tan\dfrac{\pi}{3}\right)\right) = \tan^{-1}\tan\dfrac{\pi}{3}$

$\Rightarrow \quad \beta = \dfrac{\pi}{3}$ ...(ii)

From Eqs. (i) and (ii), we get

$4\alpha = 3\beta$ **Hence proved.**

9. The domain of $\sin^{-1} x$ is $[-1, 1]$.

Therefore, $f(x) = \sin^{-1}(-x^2)$ is defined for all x satisfying

$-1 \le -x^2 \le 1$

$\Rightarrow \quad 1 \ge x^2 \ge -1$

$\Rightarrow \quad 0 \le x^2 \le 1$

$\Rightarrow \quad x^2 \le 1$

$\Rightarrow \quad x^2 - 1 \le 0$

$\Rightarrow \quad (x - 1)(x + 1) \le 0$

$\Rightarrow \quad -1 \le x \le 1$

Hence, the domain of $f(x) = \sin^{-1}(-x^2)$ is $[-1, 1]$.

10. $\tan^2(\sec^{-1} 2) + \cot^2(\text{cosec}^{-1} 3)$

$= \sec^2(\sec^{-1} 2) - 1 + \text{cosec}^2(\text{cosec}^{-1} 3) - 1$

$= [\sec\sec^{-1} 2]^2 + [\text{cosec}\ \text{cosec}^{-1} 3]^2 - 2$

$= 2^2 + 3^2 - 2$

$= 4 + 9 - 2 = 11$ **Hence proved.**

11. We have,

$f(x) = \sin^2 x + \sin^2\left(x + \dfrac{\pi}{3}\right) + \cos x \cos\left(x + \dfrac{\pi}{3}\right)$

$= \dfrac{1}{2}\left\{2\sin^2 x + 2\sin^2\left(x + \dfrac{\pi}{3}\right) + 2\cos x \cos\left(x + \dfrac{\pi}{3}\right)\right\}$

$= \dfrac{1}{2}\left\{\begin{array}{l} 1 - \cos 2x + 1 - \cos\left(2x + \dfrac{2\pi}{3}\right) \\ \qquad + \cos\left(2x + \dfrac{\pi}{3}\right) + \cos\dfrac{\pi}{3} \end{array}\right\}$

$= \dfrac{1}{2}\left\{\dfrac{5}{2} - \cos 2x - \cos\left(2x + \dfrac{2\pi}{3}\right) + \cos\left(2x + \dfrac{\pi}{3}\right)\right\}$

$= \dfrac{1}{2}\left[\dfrac{5}{2} - \left\{\cos 2x + \cos\left(2x + \dfrac{2\pi}{3}\right)\right\} + \cos\left(2x + \dfrac{\pi}{3}\right)\right]$

$= \dfrac{1}{2}\left[\dfrac{5}{2} - 2\cos\left(2x + \dfrac{\pi}{3}\right)\cos\dfrac{\pi}{3} + \cos\left(2x + \dfrac{\pi}{3}\right)\right]$

$= \dfrac{1}{2}\left[\dfrac{5}{2} - \cos\left(2x + \dfrac{\pi}{3}\right) + \cos\left(2x + \dfrac{\pi}{3}\right)\right]$

$= \dfrac{5}{4}$ for all $x \in R$

$\therefore$ For any $x \in R$, we have $gof(x) = g(f(x)) = g\left(\dfrac{5}{4}\right) = 1$

Thus, $gof(x) = 1$ for all $x \in R$. Hence, $gof : R \to R$ is a constant function. **Hence proved.**

12. We have, $x^2 + y^2 + z^2 = r^2$

Also, $\frac{yz}{xr} \times \frac{zx}{yr} = \frac{z^2}{r^2} = \frac{z^2}{x^2 + y^2 + z^2} < 1$

$\therefore \tan^{-1}\frac{yz}{xr} + \tan^{-1}\frac{xz}{yr} + \tan^{-1}\frac{xy}{zr}$

$= \tan^{-1}\left\{\frac{\frac{yz}{xr} + \frac{xz}{yr}}{1 - \frac{yz}{xr}\cdot\frac{xz}{yr}}\right\} + \tan^{-1}\frac{xy}{zr}$

$= \tan^{-1}\left\{\frac{\frac{y^2z + x^2z}{xyr}}{1 - \frac{z^2}{r^2}}\right\} + \tan^{-1}\frac{xy}{zr}$

$= \tan^{-1}\left\{\frac{z(x^2 + y^2)}{xyr} \times \frac{r^2}{r^2 - z^2}\right\} + \tan^{-1}\frac{xy}{zr}$

$= \tan^{-1}\left\{\frac{z(r^2 - z^2)}{xyr}\cdot\frac{r^2}{r^2 - z^2}\right\} + \tan^{-1}\frac{xy}{zr}$

$= \tan^{-1}\frac{zr}{xy} + \tan^{-1}\frac{xy}{zr}$

$= \tan^{-1}\frac{zr}{xy} + \cot^{-1}\frac{zr}{xy} = \frac{\pi}{2}$ **Hence proved.**

13. We have, $\sin^{-1} 6x + \sin^{-1} 6\sqrt{3}\, x = -\frac{\pi}{2}$

$\Rightarrow \sin^{-1} 6x = -\frac{\pi}{2} - \sin^{-1} 6\sqrt{3}x$

$\Rightarrow \sin(\sin^{-1} 6x) = \sin\left(-\frac{\pi}{2} - \sin^{-1} 6\sqrt{3}x\right)$

$\Rightarrow 6x = -\sin\left(\frac{\pi}{2} + \sin^{-1} 6\sqrt{3}\, x\right)$

$\Rightarrow 6x = -\cos \sin^{-1} 6\sqrt{3}\, x$

$\Rightarrow 6x = -\cos\{\cos^{-1}\sqrt{1 - (6\sqrt{3}\, x)^2}\}$

$\Rightarrow 6x = -\sqrt{1 - 108\, x^2}$

$\Rightarrow 36\, x^2 = 1 - 108\, x^2$

$\Rightarrow 144\, x^2 = 1 \Rightarrow x = \pm\frac{1}{12}$

We observe that $x = \frac{1}{12}$ does not satisfy the given equation.

Hence, $x = -\frac{1}{12}$ is the only root of the given equation.

14. We have, $y = \cot^{-1}(\sqrt{\cos x}) - \tan^{-1}(\sqrt{\cos x})$

$\Rightarrow y = \frac{\pi}{2} - \tan^{-1}(\sqrt{\cos x}) - \tan^{-1}(\sqrt{\cos x})$

$\Rightarrow y = \frac{\pi}{2} - 2\tan^{-1}(\sqrt{\cos x})$

$\Rightarrow y = \frac{\pi}{2} - \cos^{-1}\left\{\frac{1 - (\sqrt{\cos x})^2}{1 + (\sqrt{\cos x})^2}\right\}$

$\Rightarrow y = \frac{\pi}{2} - \cos^{-1}\left\{\frac{1 - \cos x}{1 + \cos x}\right\}$

$\Rightarrow y = \frac{\pi}{2} - \cos^{-1}\tan^2\frac{x}{2}$

$\Rightarrow \cos^{-1}\tan^2\frac{x}{2} = \frac{\pi}{2} - y$

$\Rightarrow \tan^2\frac{x}{2} = \cos\left(\frac{\pi}{2} - y\right)$

$\Rightarrow \tan^2\frac{x}{2} = \sin y$ **Hence proved.**

15. Using $\cos 2\alpha = \frac{1 - \tan^2\alpha}{1 + \tan^2\alpha}$ and $\sin 2\alpha = \frac{2\tan\alpha}{1 + \tan^2\alpha}$,

we get $\tan^{-1}\left(\frac{3\sin 2\alpha}{5 + 3\cos 2\alpha}\right) + \tan^{-1}\left(\frac{1}{4}\tan\alpha\right)$

$= \tan^{-1}\left(\frac{6\tan\alpha}{8 + 2\tan^2\alpha}\right) + \tan^{-1}\left(\frac{1}{4}\tan\alpha\right)$

$= \tan^{-1}\left(\frac{3\tan\alpha}{4 + \tan^2\alpha}\right) + \tan^{-1}\left(\frac{1}{4}\tan\alpha\right)$

$= \tan^{-1}\left[\frac{\frac{3\tan\alpha}{4 + \tan^2\alpha} + \frac{1}{4}\tan\alpha}{1 - \frac{3\tan\alpha}{4 + \tan^2\alpha}\cdot\frac{1}{4}\tan\alpha}\right]$

$= \tan^{-1}\left[\frac{(16 + \tan^2\alpha)\tan\alpha}{16 + \tan^2\alpha}\right]$

$= \tan^{-1}(\tan\alpha)$

$= \alpha$

16. Let $\cos^{-1}\frac{a}{b} = \theta$. Then, $\cos\theta = \frac{a}{b}$

$\therefore$ LHS $= \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) + \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$

$= \frac{1 + \tan\frac{\theta}{2}}{1 - \tan\frac{\theta}{2}} + \frac{1 - \tan\frac{\theta}{2}}{1 + \tan\frac{\theta}{2}}$

$= \frac{\left(1 + \tan\frac{\theta}{2}\right)^2 + \left(1 - \tan\frac{\theta}{2}\right)^2}{1 - \tan^2\frac{\theta}{2}} = \frac{2\left(1 + \tan^2\frac{\theta}{2}\right)}{\left(1 - \tan^2\frac{\theta}{2}\right)}$

$= \frac{2}{\cos\theta} = \frac{2b}{a}$

= RHS **Hence proved.**

17. We observe the following properties of relation R.

Reflexivity Let (a,b) be an arbitrary element of $N \times N$.

Then, $(a,b) \in N \times N$

$\Rightarrow a, b \in N \Rightarrow ab = ba$

[by commutativity of multiplication on N]

$\Rightarrow (a,b)R(a,b)$

Thus $(a,b)R(a,b)$ for all $(a,b) \in N \times N$.

So, R is reflexive on $N \times N$.

Symmetry Let $(a,b),(c,d) \in N \times N$ be such that $(a,b)R(c,d)$.

Then, $(a,b)R(c,d)$

$\Rightarrow ad = bc$

$\Rightarrow cb = da$ [by commutative of multiplication of N]

$(c,d)\ R\ (a,b)$

Thus, $(a,b)R(c,d) \Rightarrow (c,d)R(a,b)$ for all $(a,b),(c,d) \in N \times N$.

So, R is symmetric on $N \times N$.

Transitivity Let $(a,b),(c,d),(e,f) \in N \times N$

such that $(a,b)R(c,d)$ and $(c,d)R(e,f)$.

Then, $(a,b)R(c,d) \Rightarrow ad = bc$

$(c,d)R(e,f) \Rightarrow cf = de$

$\Rightarrow$ $(ad)(cf) = (bc)(de)$

$af = be \Rightarrow (a,b)R(e,f)$

Thus, $(a,b)R(c,d)$ and $(c,d)R(e,f)$

$\Rightarrow$ $(a,b)R(e,f)$ for all $(a,b),(c,d),(e,f) \in N \times N$

So, R is transitive on $N \times N$.

Hence, R being reflexive, symmetric and transitive, is an equivalence relation on $N \times N$. **Hence proved.**

Now, $[(2,6)] = \{(x,y) \in N \times N : (x,y)\ R\ (2,6)\}$

$= \{(x, y) \in N \times N : 3x = y\}$

$= \{(x, 3x) : x \in N\}$

$= \{(1, 3), (2, 6), (3, 9), (4, 12), \ldots\}$

18. Let $A(\alpha) = \begin{bmatrix} \alpha & \alpha \\ \alpha & \alpha \end{bmatrix}$, $\alpha \in R$ be the identity element in M. Then,

$A(x)\ A(\alpha) = A(x) = A(\alpha)\ A(x)$ for all $x \in R$.

$\Rightarrow$ $\begin{bmatrix} x & x \\ x & x \end{bmatrix}\begin{bmatrix} \alpha & \alpha \\ \alpha & \alpha \end{bmatrix} = \begin{bmatrix} x & x \\ x & x \end{bmatrix}$

$\Rightarrow$ $\begin{bmatrix} 2\alpha x & 2\alpha x \\ 2\alpha x & 2\alpha x \end{bmatrix} = \begin{bmatrix} x & x \\ x & x \end{bmatrix}$

$\Rightarrow$ $2\alpha x = x \Rightarrow \alpha = \frac{1}{2}$

Thus, $A\left(\frac{1}{2}\right) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ is the identity element in M.

Let $A(y) = \begin{bmatrix} y & y \\ y & y \end{bmatrix}$ be the inverse of an element $A(x) = \begin{bmatrix} x & x \\ x & x \end{bmatrix}$ in M. Then, $A(x)\ A(y) = A\left(\frac{1}{2}\right) = A(y)\ A(x)$

$\Rightarrow$ $\begin{bmatrix} x & x \\ x & x \end{bmatrix}\begin{bmatrix} y & y \\ y & y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$\Rightarrow$ $\begin{bmatrix} 2xy & 2xy \\ 2xy & 2xy \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$\Rightarrow$ $2xy = \frac{1}{2} \Rightarrow y = \frac{1}{4x}, x \neq 0$

$\therefore A(y) = A\left(\frac{1}{4x}\right) = \begin{bmatrix} \frac{1}{4x} & \frac{1}{4x} \\ \frac{1}{4x} & \frac{1}{4x} \end{bmatrix}$ is the inverse of $A(x) = \begin{bmatrix} x & x \\ x & x \end{bmatrix}$ in M.

19. For any $x \in [-1,1]$, the maximum value of $\sin^{-1} x$ is $\frac{\pi}{2}$ and it attains this value at $x = 1$.

$\therefore$ $\sin^{-1} x \le \frac{\pi}{2}, \sin^{-1} y \le \frac{\pi}{2}, \sin^{-1} z \le \frac{\pi}{2}$

for all $x, y, z \in [-1,1]$

$\Rightarrow$ $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z \le \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2}$

for all $x, y, z \in [-1,1]$

$\Rightarrow$ $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z \le \frac{3\pi}{2}$

for all $x, y, z \in [-1,1]$

$\therefore$ $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{3\pi}{2}$

$\Rightarrow$ $\sin^{-1} x = \sin^{-1} y = \sin^{-1} z = \frac{\pi}{2}$

$\Rightarrow$ $x = y = z = 1$

(i) $x^{2018} + y^{2019} + z^{2020} = (1)^{2018} + (1)^{2019} + (1)^{2020} = 3$

(ii) $x^{2016} + y^{2018} + z^{2020} - \frac{9}{x^{2016} + y^{2018} + z^{2020}}$

$= 1 + 1 + 1 - \frac{9}{1+1+1} = 3 - 3 = 0$

20. $(\sin^{-1} x)^2 + (\cos^{-1} x)^2 = (\sin^{-1} x)^2 + \left(\frac{\pi}{2} - \sin^{-1} x\right)^2$

$\left[\because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}\right]$

$= (\sin^{-1} x)^2 + \frac{\pi^2}{4} + (\sin^{-1} x)^2 - \pi(\sin^{-1} x)$

$= 2(\sin^{-1} x)^2 - \pi(\sin^{-1} x) + \frac{\pi^2}{4}$

$= 2\left[(\sin^{-1} x)^2 - \frac{\pi}{2}\sin^{-1} x + \frac{\pi^2}{8}\right]$

$= 2\left[\left(\sin^{-1} x - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{8} - \frac{\pi^2}{16}\right]$

$= 2\left[\left(\sin^{-1} x - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{16}\right]$

Now, $-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$ for all $x \in [-1,1]$

$\Rightarrow$ $-\frac{\pi}{2} - \frac{\pi}{4} \le \sin^{-1} x - \frac{\pi}{4} \le \frac{\pi}{2} - \frac{\pi}{4}$ for all $x \in [-1,1]$

$\Rightarrow$ $-\frac{3\pi}{4} \le \left(\sin^{-1} x - \frac{\pi}{4}\right) \le \frac{\pi}{4}$ for all $x \in [-1,1]$

$\Rightarrow$ $0 \le \left(\sin^{-1} x - \frac{\pi}{4}\right)^2 \le \frac{9\pi^2}{16}$

$\Rightarrow$ $\frac{\pi^2}{16} \le \left(\sin^{-1} x - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{16} \le \frac{9\pi^2}{16} + \frac{\pi^2}{16}$

$\Rightarrow$ $\frac{\pi^2}{16} \le \left(\sin^{-1} x - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{16} \le \frac{5\pi^2}{8}$

$\Rightarrow$ $\frac{\pi^2}{8} \le 2\left[\left(\sin^{-1} x - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{16}\right] \le \frac{5\pi^2}{4}$

$\Rightarrow$ $\frac{\pi^2}{8} \le (\sin^{-1} x)^2 + (\cos^{-1} x)^2 \le \frac{5\pi^2}{4}$

Hence, the greatest and least values of $(\sin^{-1} x)^2 + (\cos^{-1} x)^2$ are $\frac{5\pi^2}{4}$ and $\frac{\pi^2}{8}$, respectively.

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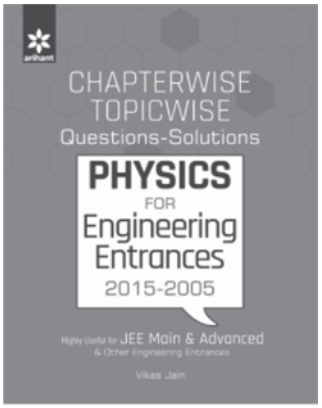

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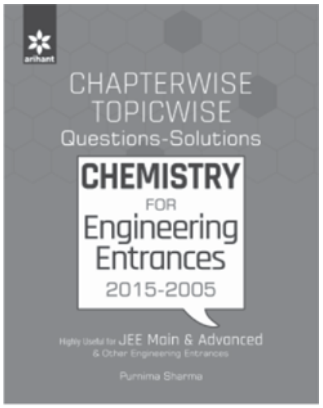

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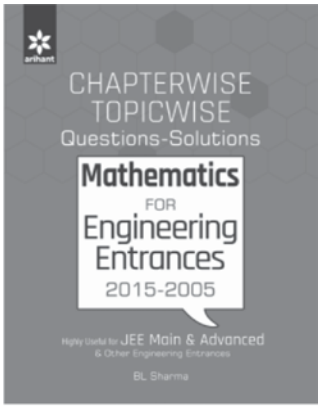

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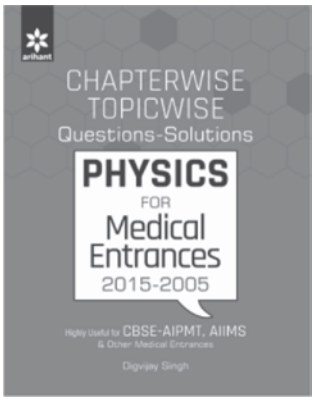

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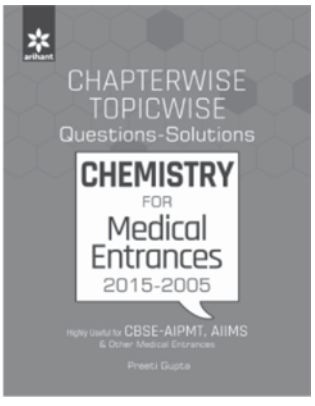

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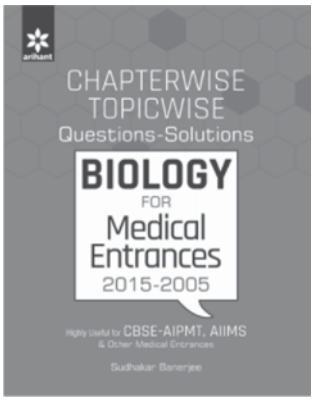

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A MUST RESOURCE FOR ENGINEERING & MEDICAL ASPIRANTS WITH ALL PRECISELY SOLVED QUESTIONS OF LAST 10 YEARS

arihant

JEE ADVANCED DRILL

30 BEST INTEGER TYPE PROBLEMS ON SEQUENCES AND SERIES

A collection of chapterwise best problems of their types

1. For $x > 1$, evaluate $\frac{x}{x+1} + \frac{x^2}{(x+1)(x^2+1)} + \frac{x^4}{(x+1)(x^2+1)(x^4+1)} + \ldots \infty$.

2. Let $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots$ and $\frac{f(x)}{1-x} = b_0 + b_1x + b_2x^2 + \ldots + b_nx^n + \ldots$. If $a_0 = 1$ and $b_1 = 3$ and $b_{10} = k^{11} - 1$, then k is

(Given that $a_0, a_1, a_2, \ldots$ are in GP).

3. Find the sum of the terms of the GP $a + ar + ar^2 + \ldots \infty$, where a is the value of x for which the function $7 + 2x \log_e 25 - 5^{x-1} - 5^{2-x}$ has the greater value and r is the limit $\lim\limits_{x \to 0} \int_0^x \frac{t^2\,dt}{\{x^2 \tan(\pi + x)\}}$.

4. Let k be a positive integer such that $36 + k$, $300 + k$, $596 + k$ are the squares of three consecutive terms of an arithmetic progression. Find $(k - 920)$.

5. If the $(m+1)$th, $(n+1)$th and $(r+1)$th terms of an AP are in GP and m, n, r are in HP, where the ratio of the common difference to the first term in the AP is $\frac{-\lambda}{n}$, then find the value of λ.

6. The largest term of the sequence $\frac{1}{503}, \frac{4}{524}, \frac{9}{581}, \frac{16}{692}, \ldots$ is t_p. Then, the value of p is

7. For the series 21, 22, 23, ... $k-1$, k, the AM and GM of the first and last numbers exist in the given series. If 'k' is a three digit number, then find the number of values which k can be attain.

8. Four different integers form an increasing AP such that one of them is the sum of squares of the remaining numbers. Find the largest number.

9. Find the HCF of the minimum non-negative value of a, b and c, given that the equation $x^4 + ax^3 + bx^2 + cx + 1 = 0$ has only real roots.

10. A sequence $a_1, a_2, a_3, \ldots, a_n$ of real numbers is such that $a_1 = 0$, $|a_2| = |a_1 + 1|$, $|a_3| = |a_2 + 1|, \ldots, |a_n|$ is equal to $|a_{n-1} + 1|$, where the arithmetic mean of $a_1, a_2, \ldots, a_n$ cannot be less than $-\lambda/\mu$, then find the value of $\lambda + \mu$.

11. Find the limiting value of the sum to n terms of the series $\frac{3}{1^2 \cdot 2^2} + \frac{5}{2^2 \cdot 3^2} + \frac{7}{3^2 \cdot 4^2} + \ldots$ as $n \to \infty$.

12. The sum to infinity of the series

$$1 + \frac{2}{3} + \frac{6}{3^2} + \frac{10}{3^3} + \frac{14}{3^4} + \ldots$$

13. Let $a_1, a_2, a_3, \ldots, a_{100}$ be an arithmetic progression with $a_1 = 3$ and $S_p = \sum\limits_{i=1}^{p} a_i, 1 \le p \le 100$. For any integer n with $1 \le n \le 20$, let $m = 5n$. If S_m/S_n does not depend on n, then find a_2.

14. Let $S_1, S_2, \ldots$ be squares such that for each $n \ge 1$, the length of a side of S_n equals the length of a diagonal of S_{n+1}. If the length of a side of S_1 is 10 cm, then for which of n, area of S_n less than 1 sq cm?

15. Let S_k, $k = 1, 2, \ldots, 100$, denotes the sum of the infinite geometric series whose first term is $\frac{k-1}{k!}$ and the common ratio is $\frac{1}{k}$. Then, find the value of $\frac{100}{100!} + \sum\limits_{k=1}^{100} |(k^2 - 3k + 1)\, S_k|$.

16. Let sum of first three terms of GP with real terms be 13/12 and their product be -1. If the absolute value of the sum of their infinite term is S, then the value of $7S$ is

17. The first term of an arithmetic progression is 1 and the sum of the first nine terms equal to 369. The first and the ninth terms of a geometric progression coincide with the first and the ninth term of the arithmetic progression. If the seventh term of the geometric progression is T_7, then the value of $T_7/9$ is

18. Let $< a_n >$ be an arithmetic progression with common difference $d\,(d \neq 0)$ and $< b_n >$ be a geometric progression with common ratio q, where q is a rational number and $0 < q < 1$. If $a_1 = d$, $b_1 = d^2$ and $\dfrac{a_1^2 + a_2^2 + a_3^2}{b_1 + b_2 + b_3}$ is a positive integer, then find $6q$.

19. Let $a_1, a_2, a_3, \ldots, a_{11}$ be real numbers satisfying $a_1 = 15$, $27 - 2a_2 > 0$ and $a_k = 2a_{k-1} - a_{k-2}$ for $k = 3, 4, \ldots, 11$. If $\dfrac{a_1^2 + a_2^2 + \ldots + a_{11}^2}{11} = 90$, then the value of $\dfrac{a_1 + a_2 + \ldots + a_{11}}{11}$ is

20. If three successive terms of a GP with common ratio $r\,(r > 1)$ form the sides of a ΔABC and $[\,]$ denotes greatest integer function, then find $|[r] + [-r]|$.

21. Let $A_n = \left(\dfrac{3}{4}\right) - \left(\dfrac{3}{4}\right)^2 + \left(\dfrac{3}{4}\right)^3 + \ldots + (-1)^{n-1}\left(\dfrac{3}{4}\right)^n$ and $B_n = 1 - A_n$. The least positive integer n_0 satisfying $B_n > A_n$ for all $n \geq n_0$, is

22. If $\sum_{r=1}^{n} r$, $\dfrac{\sqrt{10}}{3}\sum_{r=1}^{n} r^2$, $\sum_{r=1}^{n} r^3$ are in GP, then find the value of n.

23. If a is the AM of b and c and the two geometric means are G_1 and G_2 which are inserted between b and c such that $G_1^3 + G_2^3 = \lambda abc$, then λ is equal to

24. If the sum of n terms of the series
$$\frac{2n+1}{2n-1} + 3\left(\frac{2n+1}{2n-1}\right)^2 + 5\left(\frac{2n+1}{2n-1}\right)^3 + \ldots$$
is 36, then find the value of n.

25. The interior angles of a polygon are in arithmetic progression. The smallest angle is 120° and the common difference is 5°. Find the number of sides of the polygon.

26. Let $a_1, a_2, \ldots, a_{10}$ be in AP and $h_1, h_2, \ldots, h_{10}$ be in HP. If $a_1 = h_1 = 2$ and $a_{10} = h_{10} = 3$, then $a_4 h_7$ is equal to

27. For the quadratic equation $ax^2 + bx + c = 0$, $\Delta = b^2 - 4ac$ and $\alpha + \beta$, $\alpha^2 + \beta^2$, $\alpha^3 + \beta^3$ are in GP, where α, β are the roots of $ax^2 + bx + c = 0$, then find the value of $c\Delta$.

28. If H_1, H_2, H_3 and H_4 are 4 harmonic means between a and $b\,(\neq a)$, then find the value of
$$\frac{H_1 + a}{H_1 - a} + \frac{H_4 + b}{H_4 - b}.$$

29. If $\sum_{x=5}^{n+5} 4(x-3) = An^2 + Bn + C$, then find the value of $A + B - C$.

30. Let x be the arithmetic mean and y, z be the two geometric means between any two positive numbers. Then, find the value of $\dfrac{y^3 + z^3}{xyz}$.

Answers with Explanation

1. *(1)* Let $S = \dfrac{x}{x+1} + \dfrac{x^2}{(x+1)(x^2+1)} + \dfrac{x^4}{(x+1)(x^2+1)(x^4+1)} + \ldots \infty$

Then, $\dfrac{x}{x+1} = 1 - \dfrac{1}{x+1}$

$\dfrac{x^2}{(x+1)(x^2+1)} = \dfrac{1}{(x+1)} - \dfrac{1}{(x+1)(x^2+1)}$

$\dfrac{x^4}{(x+1)(x^2+1)(x^4+1)} = \dfrac{1}{(x+1)(x^2+1)} - \dfrac{1}{(x+1)(x^2+1)(x^4+1)}$

$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$

$\dfrac{x^{2^{n-1}}}{(x+1)(x^2+1)\ldots(x^{2^{n-1}}+1)} = \dfrac{1}{(x+1)(x^2+1)\ldots(x^{2^{n-2}}+1)} - \dfrac{1}{(x+1)(x^2+1)\ldots(x^{2^{n-1}}+1)}$

Now, $S_n = 1 - \dfrac{1}{(x+1)(x^2+1)\ldots(x^{2^{n-1}}+1)}$

$\therefore \quad \lim_{n \to \infty} S_n = 1$

2. *(2)* We have,

$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$

and $f(x)(1-x)^{-1} = b_0 + b_1 x + b_2 x^2 + \ldots + b_n x^n + \ldots$

$\Rightarrow \quad f(x)\{1 + x + x^2 + \ldots + x^n + \ldots \infty\} = b_0 + b_1 x + b_2 x^2 + \ldots + b_n x^n + \ldots$

Now, on comparing the coefficient of x^n from both sides, we get

$$a_0 + a_1 + a_2 + \ldots + a_{n-1} + a_n = b_n \quad \ldots(i)$$

Clearly, $b_1 = a_0 + a_1$

$\Rightarrow$ $a_1 = b_1 - a_0 = 3 - 1 = 2$ [$\because a_0 = 1$ and $b_1 = 3$]

Since, $a_0, a_1, a_2, \ldots$ are in GP.

$\therefore$ First term of this GP is 1 and common ratio is 2.

Now, consider $b_{10} = a_0 + a_1 + a_2 + \ldots + a_{10}$ [using Eq. (i)]

$= 1 + 2 + 2^2 + \ldots + 2^{10} = 2^{11} - 1$

Hence, $k = 2$

3. ***(3)*** Let $f(x) = 7 + 2x \log_e 25 - 5^{x-1} - 5^{2-x}$

$$= 7 + 4x \log_e 5 - \frac{5^x}{5} - \frac{25}{5^x}$$

$$\therefore \quad f'(x) = 4\log_e 5 - \frac{5^x}{5}\log_e 5 + 25 \cdot 5^{-x}\log_e 5$$

$$= \frac{\log_e 5}{5}[20 - 5^x + 125 \cdot 5^{-x}]$$

$$= \frac{-\log_e 5}{5}\frac{[5^{2x} - 20 \cdot 5^x - 125]}{5^x}$$

$$= \frac{-\log_e 5}{5^{x+1}}[(5^x - 25)(5^x + 5)]$$

For maximum, put $f'(x) = 0$

$\Rightarrow$ $5^x = 25 = 5^2$

$\Rightarrow$ $x = 2$

Thus, at $x = 2$, $f(x)$ is maximum, so $a = 2$.

Now, consider $r = \lim\limits_{x \to 0} \int_0^x \frac{t^2 dt}{\{x^2 \tan(\pi + x)\}}$

$$= \lim_{x \to 0} \frac{x^3}{3x^2 \tan(\pi + x)}$$

$$= \lim_{x \to 0} \frac{x}{3 \tan x} = \frac{1}{3}$$

$\Rightarrow$ $r = \frac{1}{3}$

$\therefore$ The given GP is $2 + \frac{2}{3} + \frac{2}{9} + \ldots$, whose sum is

$$\frac{2}{1 - \left(\frac{1}{3}\right)} = \frac{2}{\left(\frac{2}{3}\right)} = 3.$$

4. ***(5)*** Let the 3 consecutive terms be $a - d, a, a + d$ $(d > 0)$.

$\therefore$ $a^2 - 2ad + d^2 = 36 + k$ ...(i)

$a^2 = 300 + k$ ...(ii)

and $a^2 + 2ad + d^2 = 596 + k$ ...(iii)

Now, on subtracting Eq. (i) from Eq. (ii), we get

$d(2a - d) = 264$ ...(iv)

On subtracting Eq.(ii) from Eq. (iii), we get

$d(2a + d) = 296$ ...(v)

On subtracting Eq. (iv) from Eq. (v), we get

$2d^2 = 32$

$\Rightarrow$ $d^2 = 16$

$\Rightarrow$ $d = 4$ [$d = -4$, rejected]

Hence, from Eq. (iv),

$4(2a - 4) = 264 \Rightarrow 2a - 4 = +66$

$\Rightarrow$ $2a = 70 \Rightarrow a = 35$

$\therefore$ $k = 35^2 - 300 = 1225 - 300 = 925$

$\Rightarrow$ $k - 920 = 925 - 920 = 5$

5. ***(2)*** Let a be the first term and d be the common difference of the AP. Let x, y, z be the $(m + 1)$ th, $(n + 1)$ th and $(r + 1)$ th terms of the AP. Then, $x = a + md$, $y = a + nd$, $z = a + rd$.

Since, x, y, z are in GP.

$\therefore$ $y^2 = xz$, i.e. $(a + nd)^2 = (a + rd)(a + md)$

$\Rightarrow$ $\frac{d}{a} = \frac{r + m - 2n}{n^2 - rm}$

Now, as m, n, r are in HP.

$\therefore$ $\frac{2}{n} = \frac{1}{m} + \frac{1}{r} \Rightarrow \frac{2}{n} = \frac{m + r}{mr}$

Hence, $\frac{d}{a} = \frac{2\left(\frac{r + m}{2} - n\right)}{n\left(n - \frac{rm}{n}\right)} = \frac{2\left(\frac{mr}{n} - n\right)}{n\left(n - \frac{rm}{n}\right)}$

$= \frac{-2}{n} = -\frac{\lambda}{n} \quad \Rightarrow \quad \lambda = 2$

6. ***(7)*** Clearly, $T_n = \frac{n^2}{500 + 3n^3}$

Now, $\frac{dT_n}{dn} = \frac{(500 + 3n^3) \cdot 2n - n^2 \cdot 9n^2}{(500 + 3n^3)^2}$

$= \frac{(1000n + 6n^4 - 9n^4)}{(500 + 3n^3)^2} = \frac{(1000n - 3n^4)}{(500 + 3n^3)^2}$

$= \frac{n(1000 - 3n^3)}{(500 + 3n^3)^2}$

For maximum, put $\frac{dT_n}{dn} = 0$

$\Rightarrow$ $\frac{n(1000 - 3n^3)}{(500 + 3n^3)^2} = 0 \Rightarrow n = \left(\frac{1000}{3}\right)^{1/3}$

$\because$ $6 < \left(\frac{1000}{3}\right)^{1/3} < 7$

Hence, T_7 is the largest term and the largest term in the given sequence is $\frac{49}{1529}$.

$\therefore$ $p = 7$

7. ***(2)*** Given sequence is $21, 22, 23, \ldots, k - 1, k$.

Since, AM $= \frac{21 + k}{2}$ and GM $= \sqrt{21 \cdot k}$ exist in the given series, therefore k should be odd number and $k = 21 \cdot \lambda^2$ for some $\lambda \in N$.

Also, $100 \le k \le 999$

$\Rightarrow$ $\frac{100}{21} \le \lambda^2 \le \frac{999}{21}$

$\Rightarrow$ $4.76 \le \lambda^2 \le 47.57$

$\Rightarrow$ $\lambda = 3, 4, 5, 6$ but λ should be odd.

$\Rightarrow$ $\lambda = 3, 5 \Rightarrow k$ can assume 2 different values.

8. ***(2)*** Let AP be $a-d, a, a+d, a+2d$.

Note that a and d must be integers.

Also, this is an increasing AP. Hence, $a+2d$ is the largest, so we have

$$a+2d=(a-d)^2+a^2+(a+d)^2=3a^2+2d^2$$

$$\Rightarrow \quad 3a^2-a+2d^2-2d=0$$

As a is real.

$$\therefore \quad 1-24(d^2-d)\geq 0 \Rightarrow d^2-d-\frac{1}{24}\leq 0$$

$$\Rightarrow \quad \left(d-\frac{1}{2}\right)^2\leq \frac{3}{24}\Rightarrow \frac{1}{2}-\frac{\sqrt{7}}{2\sqrt{6}}\leq d\leq \frac{1}{2}+\frac{\sqrt{7}}{2\sqrt{6}}$$

As d is an integer, so $d=0, 1$

But $d\neq 0$, therefore $d=1$

Thus, $3a^2-a=0\Rightarrow a=0$ or $a=\frac{1}{3}$

As a is an integer $\Rightarrow a=0$

Hence, required number is 2.

9. ***(2)*** Let $f(x)=x^4+ax^3+bx^2+cx+1$

As a, b and c are non-negative, no root of the equation $f(x)=0$ can be positive. Further as $f(0)\neq 0$, all the roots of the equation, say x_1, x_2, x_3 and x_4 are negative.

We have, $\Sigma x_1=-a, \Sigma x_1x_2=b, \Sigma x_1x_2x_3=-c$ and $x_1x_2x_3x_4=1$

Using AM $\geq$ GM for positive numbers $-x_1, -x_2, -x_3$ and $-x_4$, we get $\frac{a}{4}\geq 1\Rightarrow a\geq 4$

Using AM $\geq$ GM for positive numbers x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4 and x_3x_4, we get $\frac{b}{6}\geq 1\Rightarrow b\geq 6$

Finally, using AM $\geq$ GM for positive numbers $-x_1x_2x_3$, $-x_1x_2x_4$, $-x_1x_3x_4$ and $-x_2x_3x_4$, we get $\frac{c}{4}\geq 1\Rightarrow c\geq 4$

$\therefore$ HCF of $\{a, b, c\}$ = HCF of $\{4, 6, 4\}=2$

10. ***(3)*** Let us add one more number, a_{n+1} to the given sequence.

The number a_{n+1} is such that $|a_{n+1}|=|a_n+1|$.

On squaring all the numbers, we have

$$a_1^2=0$$
$$a_2^2=a_1^2+2a_1+1$$
$$a_3^2=a_2^2+2a_2+1$$
$$a_4^2=a_3^2+2a_3+1$$
$$\vdots \quad \vdots \quad \vdots \quad \vdots$$
$$a_n^2=a_{n-1}^2+2a_{n-1}+1$$
$$a_{n+1}^2=a_n^2+2a_n+1$$

On adding the above equalities, we get

$$a_1^2+a_2^2+\ldots+a_n^2+a_{n+1}^2$$
$$=a_1^2+a_2^2+\ldots+a_n^2+2(a_1+a_2+\ldots+a_n)+n$$

$$\Rightarrow \quad 2(a_1+a_2+\ldots+a_n)=-n+a_{n+1}^2\geq -n$$

$$\Rightarrow \quad \frac{a_1+a_2+\ldots+a_n}{n}\geq -\frac{1}{2}=\frac{-\lambda}{\mu}$$

$$\therefore \quad \lambda+\mu=3$$

11. ***(1)*** Let T_r be the rth term of the given series. Then,

$$T_r=\frac{(2r+1)}{r^2(r+1)^2}, r=1, 2, 3, \ldots$$

$$\Rightarrow \quad T_r=\left\{\frac{1}{r^2}-\frac{1}{(r+1)^2}\right\}, r=1, 2, 3, \ldots$$

$\therefore$ Required sum $=\lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}T_r$

$$\Rightarrow \text{Required sum}=\lim_{n\to\infty}\sum_{r=1}^{n}\left\{\frac{1}{r^2}-\frac{1}{(r+1)^2}\right\}$$

$$\Rightarrow \text{Required sum}=\lim_{n\to\infty}\left\{1-\frac{1}{(n+1)^2}\right\}=1$$

12. ***(3)*** Let S be the sum of the given series, i.e.

$$S=1+\frac{2}{3}+\frac{6}{3^2}+\frac{10}{3^3}+\frac{14}{3^4}+\ldots$$

$$(S-1)=\frac{2}{3}+\frac{6}{3^2}+\frac{10}{3^3}+\frac{14}{3^4}+\ldots \quad \ldots(i)$$

$$(S-1)\times\frac{1}{3}=\frac{2}{3^2}+\frac{6}{3^3}+\frac{10}{3^4}+\ldots \quad \ldots(ii)$$

On subtracting Eq. (ii) from Eq. (i), we get

$$\frac{2}{3}(S-1)=\frac{2}{3}+\frac{4}{3^2}+\frac{4}{3^3}+\frac{4}{3^4}+\ldots$$

$$\Rightarrow \quad \frac{2}{3}(S-1)=\frac{2}{3}+\frac{\frac{4}{3^2}}{1-\frac{1}{3}}$$

$$\Rightarrow \quad \frac{2}{3}(S-1)=\frac{2}{3}+\frac{2}{3}$$

$$\Rightarrow \quad S=3$$

13. ***(9)*** Since, $a_1, a_2, a_3, \ldots, a_{100}$ is an AP with $a_1=3$ and $S_p=\sum\limits_{i=1}^{p}a_i$

$$\therefore \quad \frac{S_m}{S_n}=\frac{S_{5n}}{S_n}=\frac{\frac{5n}{2}\{2\times 3+(5n-1)\times d\}}{\frac{n}{2}\{2\times 3+(n-1)\times d\}}$$

$$=\frac{5(5nd+6-d)}{(nd+6-d)}$$

Clearly, it will be independent of n, if

$$6-d=0, \text{ i.e. } d=6$$

$$\therefore \quad a_2=a_1+d=3+6=9$$

14. ***(8)*** We have, length of a side of S_n = length of a diagonal of S_{n+1}

Length of a side of $S_n=\sqrt{2}$ (Length of a side of S_{n+1})

$$\Rightarrow \quad \frac{\text{Length of a side of } S_{n+1}}{\text{Length of side of } S_n}=\frac{1}{\sqrt{2}} \text{ for all } n\geq 1$$

Sides of $S_1, S_2, \ldots, S_n$ form a GP with common ratio $\frac{1}{\sqrt{2}}$ and first term 10.

$$\therefore \text{ Length of the side of } S_n=10\left(\frac{1}{\sqrt{2}}\right)^{n-1}=\frac{10}{2^{\frac{n-1}{2}}}$$

Area of S_n = (Side)^2 = $\left(\frac{10}{2^{\frac{n-1}{2}}}\right)^2 = \frac{100}{2^{n-1}}$

Now, area of $S_n < 1$

$\Rightarrow \quad \frac{100}{2^{n-1}} < 1$

$\Rightarrow \quad 2^{n-1} > 100$

$\Rightarrow \quad n - 1 \geq 7 \Rightarrow n \geq 8$

15. ***(3)*** We have,

$$S_k = \frac{\frac{k-1}{k!}}{1-\frac{1}{k}} = \frac{1}{(k-1)!}$$

$$\therefore \quad \sum_{k=2}^{100} \left| (k^2 - 3k + 1) S_k \right|$$

$$= \sum_{k=2}^{100} \left| \frac{\{(k-1)^2 - k\}}{(k-1)!} \right| = \sum_{k=2}^{100} \left| \frac{k-1}{(k-2)!} - \frac{k}{(k-1)!} \right|$$

$$= \sum_{k=2}^{100} \left| \frac{(k-2)+1}{(k-2)!} - \frac{(k-1)+1}{(k-1)!} \right|$$

$$= \sum_{k=2}^{100} \left| \frac{1}{(k-3)!} - \frac{1}{(k-2)!} + \frac{1}{(k-2)!} - \frac{1}{(k-1)!} \right|$$

$$= \sum_{k=3}^{100} \left\{ \frac{1}{(k-3)!} - \frac{1}{(k-2)!} \right\} + \sum_{k=2}^{100} \left\{ \frac{1}{(k-2)!} - \frac{1}{(k-1)!} \right\}$$

$$= \left\{ \left(1 - \frac{1}{1!}\right) + \left(\frac{1}{1!} - \frac{1}{2!}\right) + \left(\frac{1}{2!} - \frac{1}{3!}\right) + \ldots + \left(\frac{1}{97!} - \frac{1}{98!}\right) \right\}$$

$$+ \left\{ \left(1 - \frac{1}{1!}\right) + \left(\frac{1}{1!} - \frac{1}{2!}\right) + \left(\frac{1}{2!} - \frac{1}{3!}\right) + \ldots + \left(\frac{1}{98!} - \frac{1}{99!}\right) \right\}$$

$$= \left(1 - \frac{1}{98!}\right) + \left(1 - \frac{1}{99!}\right) = 2 - \frac{1}{98!} - \frac{1}{99!} = 2 - \frac{100}{99!}$$

Hence, $\frac{(100)}{100!} + \sum_{k=1}^{100} |(k^2 - 3k + 1) S_k|$

$$= \frac{100}{99!} + S_1 + \sum_{k=2}^{100} |(k^2 - 3k + 1) S_k|$$

$$= \frac{100}{99!} + 1 + 2 - \frac{100}{99!} = 3$$

16. ***(4)*** Let $\frac{a}{r}, a, ar$ be three terms in GP.

$\therefore$ Product of terms = $a^3 = -1$ [given]

$\Rightarrow \quad a = -1$

Now, sum of terms = $\frac{a}{r} + a + ar = \frac{13}{12}$ [given]

$$\frac{-1}{r} - 1 - r = \frac{13}{12} \qquad [\because a = -1]$$

$\Rightarrow \quad 12r^2 + 25r + 12 = 0$

$\Rightarrow \quad (3r + 4)(4r + 3) = 0$

$\Rightarrow \quad r = -\frac{4}{3}, -\frac{3}{4}$

But $\quad r \neq -\frac{4}{3}$

$\therefore \quad |S| = \left|\frac{a}{1-r}\right| = \left|\frac{-1}{1-\left(-\frac{3}{4}\right)}\right|$

$$= \left|\frac{-1}{1+\frac{3}{4}}\right| = \frac{-4}{7} = \frac{4}{7}$$

Hence, $7S = 4$

17. ***(3)*** $369 = \frac{9}{2}[2 + (9-1)d]$

$\Rightarrow \quad 82 = 2 + 8d$

$\Rightarrow \quad d = 10$

Now, $ar^8 = a + 8d = 1 + 8 \times 10 = 81$

$\Rightarrow \quad r^8 = 81$ $[\because a = 1]$

$\Rightarrow \quad r = \sqrt{3}$

Now, $T_7 = ar^{(7-1)} = 1 \times (\sqrt{3})^6 = 27$

Hence, $T_7 / 9 = 3$

18. ***(3)*** $\because a_k = kd$ and $b_k = d^2 q^{k-1}, \forall\ k \in N$

Now, $\frac{a_1^2 + a_2^2 + a_3^2}{b_1 + b_2 + b_3} = \frac{d^2(1^2 + 2^2 + 3^2)}{d^2(1 + q + q^2)}$

$$= \frac{14}{1 + q + q^2}$$

As $\frac{14}{1 + q + q^2}$

is a positive integer, say m, we get

$$1 + q + q^2 = \frac{14}{m}$$

$$q = -\frac{1}{2} + \sqrt{\frac{14}{m} - \frac{3}{4}} \qquad [\because 0 < q < 1]$$

As $0 < q < 1$, therefore we have

$$\frac{1}{2} < \sqrt{\frac{14}{m} - \frac{3}{4}} < \frac{3}{2}$$

$\Rightarrow \quad \frac{1}{4} < \frac{14}{m} - \frac{3}{4} < \frac{9}{4}$

$\Rightarrow \quad 1 < \frac{14}{m} < 3$

$\Rightarrow \quad 5 \leq m \leq 13$

Note that $m = 8$ makes $\frac{14}{m} - \frac{3}{4}$ a perfect square of a rational number.

For $m = 8$, $q = \frac{1}{2} \Rightarrow 6q = 3$

19. ***(0)*** We have,

$a_k = 2a_{k-1} - a_{k-2}$ for 3, 4, ..., 11

$a_k - a_{k-1} = a_{k-1} - a_{k-2}$ for $k = 3, 4, \ldots, 11$

$a_1, a_2, a_3, \ldots, a_{11}$ are in AP.

Let d be the common difference of the AP.

$a_k = 15 + (k-1)d$, $k = 1, 2, \ldots, 11$

$$\frac{a_1^2 + a_2^2 + \ldots + a_{11}^2}{11} = 90$$

$$\sum_{k=1}^{11} a_k^2 = 990 \Rightarrow \sum_{k=1}^{11} \{15 + (k-1)d\}^2 = 990$$

$$\Rightarrow \sum_{k=1}^{11} \{225 + 30(k-1)d + (k-1)^2 d^2\} = 990$$

$$\Rightarrow 225 \times 11 + 30d \times \frac{10 \times 11}{2} + \frac{10 \times 11 \times 21}{6} d^2 = 990$$

$$\Rightarrow 225 + 150d + 35d^2 = 90$$

$$\Rightarrow 35d^2 + 150d + 135 = 0$$

$$\Rightarrow 7d^2 + 30d + 27 = 0$$

$$\Rightarrow (7d + 9)(d + 3) = 0$$

$$\Rightarrow d = -3, \frac{-9}{7}$$

It is given that $a_2 < \frac{27}{2}$. Therefore, $d = -3$

Now, $\frac{a_1 + a_2 + \ldots + a_{11}}{11} = \frac{1}{2}(a_1 + a_{11})$

$$= \frac{1}{2}(15 + 15 + 10d) = 0$$

20. ***(1)*** Let the sides of a triangle be a, ar, ar^2. Since, $r > 1$. Therefore, ar^2 is the largest side.

$\therefore$ $a + ar > ar^2$

$ar^2 - a - ar < 0$

$r^2 - r - 1 < 0$

$\frac{1-\sqrt{5}}{2} < r < \frac{1+\sqrt{5}}{2}$

$1 < r < \frac{1+\sqrt{5}}{2} \Rightarrow [r] = 1$ $[\because r > 1]$

Also, $-\left(\frac{1+\sqrt{5}}{2}\right) < -r < -1$

$\Rightarrow$ $[-r] = -2$

Hence, $[r] + [-r] = 1 - 2 = -1$

and $|[r] + [-r]| = 1$

21. ***(7)*** We have,

$$A_n = \frac{3}{4} - \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \ldots + (-1)^{n-1}\left(\frac{3}{4}\right)^n$$

$$A_n = -\left(\frac{-3}{4}\right) - \left(-\frac{3}{4}\right)^2 - \left(-\frac{3}{4}\right)^3 + \ldots + (-1)^n\left(-\frac{3}{4}\right)^n$$

$$A_n = -\left[\left(-\frac{3}{4}\right)\left\{\frac{1-\left(-\frac{3}{4}\right)^n}{1-\left(-\frac{3}{4}\right)}\right\}\right]$$

$$= \frac{3}{7}\left\{1 - \left(-\frac{3}{4}\right)^n\right\}$$

Now, $B_n > A_n$

$1 - A_n > A_n$ $[\because B_n = 1 - A_n]$

$A_n < \frac{1}{2}$, $\frac{3}{7}\left\{1 - \left(-\frac{3}{4}\right)^n\right\} < \frac{1}{2}$

$$1 - \left(-\frac{3}{4}\right)^n < \frac{7}{6}$$

$$\Rightarrow \left(-\frac{3}{4}\right)^n > -\frac{1}{6}$$

Clearly, it is true for all even natural numbers. If n is an odd natural numbers, then

$$\left(-\frac{3}{4}\right)^n > -\frac{1}{6}$$

$$\Rightarrow -6 \times 3^n > -4^n$$

$$\Rightarrow 3^{n+1} < 2^{2n-1}$$

This is true for all $n \geq 7$. Hence, $n_0 = 7$

22. ***(4)*** We have,

$$\frac{10}{9}\left(\sum_{r=1}^{n} r^2\right)^2 = \left(\sum_{r=1}^{n} r\right)\left(\sum_{r=1}^{n} r^3\right)$$

$$\Rightarrow \frac{10}{9}\left\{\frac{n(n+1)(2n+1)}{6}\right\}^2 = \frac{n(n+1)}{2} \times \left\{\frac{n(n+1)}{2}\right\}^2$$

$$\Rightarrow \frac{10}{9}\left(\frac{2n+1}{3}\right)^2 = \frac{n(n+1)}{2}$$

Clearly, $n = 4$ satisfies this equation.

23. ***(2)*** It is given that a is the AM of b and c.

$\therefore$ $a = \frac{b+c}{2}$

$\Rightarrow$ $b + c = 2a$ ...(i)

Since, G_1 and G_2 are two geometric means between b and c. Therefore b, G_1, G_2, c is a G P with common ratio r given by

$$r = \left(\frac{c}{b}\right)^{1/3}$$

$\therefore$ $G_1 = br = b\left(\frac{c}{b}\right)^{1/3} = c^{1/3}b^{2/3}$

$G_2 = br^2 = b\left(\frac{c}{b}\right)^{2/3} = b^{1/3}c^{2/3}$

$G_1^3 = b^2c$ and $G_2^3 = bc^2$

$G_1^3 + G_2^3 = b^2c + bc^2 = bc(b + c) = 2abc$

[using Eq. (i)]

$\therefore$ $G_1^3 + G_2^3 = \lambda abc$

$\Rightarrow$ $\lambda = 2$

24. ***(4)*** Let $\frac{2n+1}{2n-1} = r$. Then, the given series is

$S = r + 3r^2 + 5r^3 + 7r^4 + \ldots + (2n-1)r^n$ ...(i)

$rS = r^2 + 3r^3 + 5r^4 + \ldots + (2n-3)r^n + (2n-1)r^{n+1}$ ...(ii)

On subtracting Eq. (ii) from Eq. (i), we get

$$(1-r)S = r + 2r^2 + 2r^3 + \ldots + 2r^n - (2n-1)r^{n+1}$$

$$36(1-r) = r + \frac{2r^2(1-r^{n-1})}{1-r} - (2n-1)r^{n+1}$$

$$36(1-r)^2 = r - r^2 + 2r^2 - 2r^{n+1} - (2n-1)r^{n+1} + (2n-1)r^{n+2}$$

$$= r + r^2 - (2n+1)r^{n+1} + (2n-1)r^{n+2}$$

$$= r + r^2 - (2n-1)\left[\frac{2n+1}{2n-1}r^{n+1} - r^{n+2}\right]$$

$$= r + r^2 - (2n-1)[rr^{n+1} - r^{n+2}]$$

$$= r(1+r)$$

$$\Rightarrow \quad 36\left[1 - \frac{2n+1}{2n-1}\right]^2 = \frac{2n+1}{2n-1}\left[1 + \frac{2n+1}{2n-1}\right]$$

$$\Rightarrow \quad 36\left[\frac{-2}{2n-1}\right]^2 = \frac{2n+1}{2n-1}\left[\frac{4n}{2n-1}\right]$$

$$\Rightarrow \quad 36 = n(2n+1)$$

$$\Rightarrow \quad n = 4$$

25. ***(9)*** Let there be n sides in the polygon. Then, by geometry, sum of all n interior angles of polygon is $(n-2)\times 180^\circ$. Also the angles are in AP with the smallest angle 120° and common difference 5°. Therefore, sum of all interior angles of polygon is

$$\frac{n}{2}[2\times 120 + (n-1)\times 5]$$

Thus, we must have $\frac{n}{2}[2\times 120 + (n-1)\times 5] = (n-2)\times 180$

$$\Rightarrow \quad \frac{n}{2}[5n + 235] = (n-2)\times 180$$

$$\Rightarrow \quad 5n^2 + 235n = 360n - 720$$

$$\Rightarrow \quad 5n^2 - 125n + 720 = 0$$

$$\Rightarrow \quad n^2 - 25n + 144 = 0$$

$$\Rightarrow \quad (n-16)(n-9) = 0$$

$$\Rightarrow \quad n = 16, 9$$

But if $n = 16$, then 10th angle $= 120 + 15\times 5 = 195 > 180^\circ$ which is not possible. Hence, $n = 9$

26. ***(6)*** Given, $a_1 = h_1 = 2$ and $a_{10} = h_{10} = 3$

Now, consider $3 = a_{10} = 2 + 9d \Rightarrow d = \frac{1}{9}$

$$\therefore \quad a_4 = 2 + 3d = \frac{7}{3}$$

Also, $\quad 3 = h_{10} \Rightarrow \frac{1}{3} = \frac{1}{h_{10}} = \frac{1}{2} + 9D$

$$\Rightarrow \quad D = -\frac{1}{54}$$

$$\Rightarrow \quad \frac{1}{h_7} = \frac{1}{2} + 6D = \frac{1}{2} - \frac{1}{9} = \frac{7}{18}$$

$$\therefore \quad a_4 h_7 = \frac{7}{3}\times\frac{18}{7} = 6$$

27. ***(0)*** $\alpha+\beta, \alpha^2+\beta^2, \alpha^3+\beta^3$ are in GP.

Hence, $\quad (\alpha^2+\beta^2)^2 = (\alpha+\beta)(\alpha^3+\beta^3)$

$$\Rightarrow \quad \alpha^4 + \beta^4 + 2\alpha^2\beta^2 = \alpha^4 + \beta^4 + \alpha\beta^3 + \beta\alpha^3$$

$$\Rightarrow \quad \alpha\beta(\alpha^2 + \beta^2 - 2\alpha\beta) = 0$$

$$\Rightarrow \quad \alpha\beta(\alpha-\beta)^2 = 0$$

$$\Rightarrow \quad \alpha\beta = 0 \text{ or } \alpha - \beta = 0$$

$$\Rightarrow \quad \frac{c}{a} = 0 \text{ or } \alpha = \beta$$

$$\Rightarrow \quad c = 0 \text{ or } \Delta = 0 \qquad \text{[equal roots]}$$

$$\therefore \quad c\Delta = 0$$

28. ***(8)*** Clearly, a, H_1, H_2, H_3, H_4, b are in HP.

$$\Rightarrow \quad \frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \frac{1}{H_4}, \frac{1}{b} \text{ are in AP.}$$

Let d be the common difference of this AP, then

$$\frac{1}{b} = \frac{1}{a} + (6-1)d \Rightarrow \frac{1}{b} = \frac{1}{a} + 5d$$

and $\quad \frac{1}{H_4} - \frac{1}{H_1} = (4-1)d \Rightarrow \frac{1}{H_4} - \frac{1}{H_1} = 3d$

Now, $\quad \frac{H_1 + a}{H_1 - a} = \frac{\frac{1}{a} + \frac{1}{H_1}}{\frac{1}{a} - \frac{1}{H_1}} = \frac{\frac{1}{a} + \frac{1}{H_1}}{-d}$

and $\quad \frac{H_4 + b}{H_4 - b} = \frac{\frac{1}{b} + \frac{1}{H_4}}{\frac{1}{b} - \frac{1}{H_4}} = \frac{\frac{1}{b} + \frac{1}{H_4}}{d}$

$$\therefore \quad \frac{H_1 + a}{H_1 - a} + \frac{H_4 + b}{H_4 - b} = \frac{1}{d}\left[\frac{1}{b} - \frac{1}{a} + \frac{1}{H_4} - \frac{1}{H_1}\right]$$

$$= \frac{1}{d}[5d + 3d] = \frac{1}{d}\times 8d = 8$$

29. ***(4)*** $\sum_{x=5}^{n+5} 4(x-3) = An^2 + Bn + C$

$$\Rightarrow \quad \sum_{x=1}^{n+1} 4(x+1) = An^2 + Bn + C$$

$$\Rightarrow \quad 4\left\{\sum_{x=1}^{n+1} x + (n+1)\right\} = An^2 + Bn + C$$

$$\Rightarrow \quad 4\left\{\frac{(n+1)(n+2)}{2} + (n+1)\right\} = An^2 + Bn + C$$

$$\Rightarrow \quad 2\{n^2 + 5n + 4\} = An^2 + Bn + C$$

$$\Rightarrow \quad A = 2, B = 10, C = 8$$

$$\therefore \quad A + B - C = 2 + 10 - 8 = 4$$

30. ***(2)*** Given that x is the AM between a and b and y, z are the GM between a and b, where a and b are positive numbers. Then, a, x, b are in AP. So,

$$x = \frac{a+b}{2}$$

a, y, z, b are in GP.

So, $\quad y = ar$ and $z = ar^2$, where $r = 3\sqrt{\frac{b}{a}}$.

Also, $\quad yz = ab$

Now, $\quad \frac{y^3 + z^3}{xyz} = \frac{a^3r^3 + a^3r^6}{ab\left(\frac{a+b}{2}\right)}$

$$= \frac{a^3\times\frac{b}{a} + a^3\times\frac{b^2}{a^2}}{ab\left(\frac{a+b}{2}\right)}$$

$$= \frac{2(a^2b + ab^2)}{a^2b + ab^2} = 2$$

PERSONALITIES SPECTRUM

"PERSONALITIES COLUMN IS AN EFFORT TO INTRODUCE GREAT SCIENTISTS, WHOSE CONTRIBUTION IN MATHS IS UNFORGETTABLE. IN THIS ISSUE, WE WILL INTRODUCE SRINIVASA RAMANUJAN".

SRINIVASA RAMANUJAN (1887-1920)

LIFE HISTORY

Srinivasa Ramanujan (22 December 1887-26 April 1920) was an Indian mathematician. He was born in Erode, Madras Presidency (now Tamil Nadu) in a Brahmin family. Ramanujan was something of a miracle working with numbers. He was a "MAN OF NUMBERS". By the age of 13, he mastered the book on advanced trigonometry by SL Loney and discovered sophisticated theorems on his own. When he was 16, he studied the contents of the book titled "A Synopsis of Elementary Results in Pure and Applied Mathematics" (a collection of 5000 theorems). This book is generally acknowledged as a key element in awakening the genius of Ramanujan.

He secured scholarship to the university of Madras but lost it the following year because he neglected all other studies in pursuit of Mathematics and start continued to pursue independent research in Mathematics. Too poor to afford notebooks and paper, he would work out theorems on a large slate and erase them with his elbow. He had an extraordinary ability to see patterns. While he rarely proved his results, he left a host of evaluations of sums and integrals. He was especially expert in a part of number theory called modular forms. Unable to find a good job without a degree, he began working as a clerk. In 1911, he published the first of his papers in the "Journal of the Indian Mathematical Society." His genius slowly gained recognition and in 1913, he began a correspondence with the British mathematician Godprey H. Hardy that led to a grant from Trinity college, Cambridge. In England, GH Hardy tutored him and collaborated with him in some research. He made further advances there especially in the partition of numbers. His papers were published in English and European journals. It is said by GH Hardy that Ramanujan's discoveries are unusually rich and there is often more to them than initially meets the eye. As a by-product, new directions of research were opened up. Unfortunately, he died at the age of 32, while he was still at the peak of his powers.

ACHIEVEMENTS

He was elected a fellow of the Royal Society in 1918, as one of the youngest fellows in the history of the Royal Society. He was elected for his investigation in elliptic functions and the theory of numbers. The same year, he was also elected a fellow of Trinity college —the first Indian to be so honoured.

CONTRIBUTION

Ramanujan's knowledge of Mathematics was startling. Although he was almost completely unaware of modern developments in Mathematics, his mastery of continued fractions was unequated by any living mathematician. What made his achievements really extraordinary was the fact that he received almost no formal training in pure Mathematics and started working on his own Mathematical research in isolation. Some of his contributions are given below

(i) His greatest findings is his work on partitions. Basically, he intuited a formula that could determine how many ways there were to add upto a certain number.

(ii) He suggested the formula for infinite series of π.

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26390k)}{(k!)^4 (396)^{4k}}$$

His series for π converges extraordinarily rapidly (exponentially) and forms the basis of some of the fastest algos currently used to calculate π.

(iii) Ramanujan was "A MAN OF INTUITION". As recently as 2012, researchers were finally proving and finding applications for mathematical theories. He wrote on his deathbed, which he claimed came to him in dreams. Some of this work is being used in the study of black holes, which were decades away from being theorized when he died in 1920.

(iv) He had independently developed and investigated the Bernoulli numbers and had calculated the Euler-Mascherani constant upto 15 decimal places.

(v) He left behind three notebooks and a sheaf of pages (also called the "Lost Notebook") containing many unpublished results that mathematicians continued to verify long after his death.

HONOUR

(i) Ramanujan's birthday, 22 December, is celebrated as 'State IT Day' in his home state Tamil Nadu.

(ii) On the 125th anniversary of his birth, India declared his birthday as 'National Mathematics Day.'

(iii) The man who knew infinity makes its way to theatres. It's a real life superhero movie. The film written and directed by Gnana Rajasekaran. It is a film based on the book "The man who knew Infinity : A life of the Genius Ramanujan" by Robert Kanigel. In the film, Ramanujan is portrayed by British actor Dev Patel.

MATHEMATICS @ SPECTRUM FOR YOU

In this Section, we invite our readers to send their problem and we will try to provide the best possible solutions given by our expert panel.

Q.61 Sir, I am unable to find the solution of question given below. Please give me detailed solution of it.

'The line $x + y = a$ meets X and Y-axes at A and B respectively. A ΔAMN is inscribed in the ΔOAB, O being the origin with right angle at N. M and N lie respectively on OB and AB. If the area of ΔAMN is 3/8 of the area of the ΔOAB. Then, find AN/BN'.

[Saroj, Ranchi]

- Dear Saroj, here I am giving the detailed solution of your question.

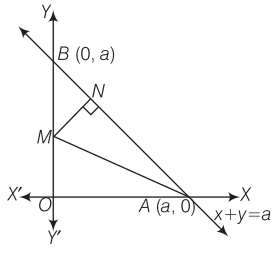
Let $\frac{AN}{BN} = \lambda$, then the coordinates of N are $\left(\frac{a}{1+\lambda}, \frac{\lambda a}{1+\lambda}\right)$

where, $(a, 0)$ and $(0, a)$ are the coordinates of A and B, respectively.

Now, equation of MN perpendicular to AB is

$$y - \frac{\lambda a}{1+\lambda} = x - \frac{a}{1+\lambda} \quad \left[\because \text{slope of } MN = \frac{-1}{\text{slope of } AB}\right]$$

$$\Rightarrow \quad x - y = \left(\frac{1-\lambda}{1+\lambda}\right)a$$



So, the coordinates of M are $\left(0, \left\{\frac{\lambda-1}{\lambda+1}\right\}a\right)$.

Now, the area of the ΔAMN

$$= \frac{1}{2}\left|\left[a\left(\frac{-a}{\lambda+1}\right) + \left\{\frac{1-\lambda}{(1+\lambda)^2}\right\}a^2\right]\right| = \frac{\lambda a^2}{(1+\lambda)^2}$$

Also, area of the $\Delta OAB = a^2/2$

According to the given condition, we have $\frac{\lambda a^2}{(1+\lambda)^2} = \frac{3}{8}\cdot\frac{1}{2}a^2$

$$\Rightarrow \quad 3\lambda^2 - 10\lambda + 3 = 0 \Rightarrow \lambda = 3 \text{ or } \lambda = \frac{1}{3}$$

For $\lambda = \frac{1}{3}$, M lies outside the segment OB and hence the required value of λ is 3 .

Q2 Sir, I am unable to crack the answer of question given below. Please provide me the detailed solution of it.

'The difference between the greatest and least value of the function $f(x)=\cos x+\frac{1}{2}\cos 2x-\frac{1}{3}\cos 3x$ is

(a) $\frac{2}{3}$ (b) $\frac{8}{7}$ (c) $\frac{9}{4}$ (d) $\frac{3}{8}$

[Anil, Kolkata]

- Dear Anil, here the given function is periodic with period 2π. So, the difference between the greatest and least value of the function is the difference between these value on the interval $[0, 2\pi]$.

Clearly, $f'(x)=-(\sin x+\sin 2x-\sin 3x)$

$=-4\sin x\sin(3x/2)\sin(x/2)$

Hence, $x=0, 2\pi/3, \pi, \frac{4\pi}{3}$ and 2π are the critical points. Also,

$$f(0)=1+\frac{1}{2}-\frac{1}{3}=\frac{7}{6},\ f(2\pi/3)=-\frac{13}{12}$$

$$f(\pi)=-\frac{1}{6},\ f\left(\frac{4\pi}{3}\right)=\frac{-13}{12} \text{ and } f(2\pi)=\frac{7}{6}.$$

Hence, the greatest value is $\frac{7}{6}$ and the least value is $\frac{-13}{12}$.

Thus, the difference is $\frac{7}{6}-\left(\frac{-13}{12}\right)=\frac{27}{12}=\frac{9}{4}$

Q3 Along a road, lie an odd number of stones placed at intervals of 10 m. These stones have to be assembled around the middle stone. A person can carry only one stone at a time. A man carried out the job starting with the stone in the middle, carrying stones in succession, thereby covering a distance of 4.8 km. Then, the number of stones is

(a) 35 (b) 29 (c) 31 (d) 33

Sir, I am facing difficulties to solve the above question. Please give me, its detailed solution. [Sumit, Delhi]

- Dear Sumit, I am giving the best solution of your question, which would be easiest to understand.

Let there be $2n+1$ stones, i.e. n stones on each side of the middle stone. The man will cover 20 m, to pick up the first stone and return, 40 m for the second stone and so on. So, he covers $(n/2)\{2\times 20+(n-1)20\}$

$=10n(n+1)$ m to pick up the stones on one side and hence $20n(n+1)$ m, to pick up all the stones.

$$20n(n+1)=4800\Rightarrow n=15$$

$\therefore$ There are $(2n+1)=31$ stones

Q4 Sir, I am facing problems in finding the answer of question given below. Please give me detailed solution of it.

'The integers from 1 to 1000 are written in order around a circle. Starting at 1, every fifteenth number is marked. (i.e. 1, 16, 31, etc). This process is continued until a number is reached which has already been marked, then find total number of unmarked number'. [Shajid, UP]

- Dear Shajid, here I am giving the detailed solution of your question.

In first round all the integers, which leaves the remainder 1 when divided by 15, will be marked.

Last number of this category is 991

Next number to be marked is $(991+15-1000)=6$

Again, in second round all the integers which leaves the remainder 6 when divided by 15, will be marked.

Last number of this category is 996.

Next number to be marked is $(996+15-1000)=11$. Thus, in third round all the integers which leaves the remainder 11 when divided by 15, will be marked and last number of this category is 986.

Next number to be marked is $986+15-1000=1$ which is already been marked.

Hence, the number of the form $15\lambda+1, 15\lambda+6, 15\lambda+11$ are marked. All these numbers can also be put in the form of $5\lambda_1+1$ $(\lambda_1\geq 0)$

Hence, total number of marked numbers $=1+\left[\frac{999}{5}\right]=200$

$\therefore$ Unmarked numbers $=800$

Q5 'The angle of elevation of the top of vertical tower from a point 'A' on the horizontal ground is found to be $\frac{\pi}{4}$. From 'A', a man walks 10 m up a path sloping at a angle $\frac{\pi}{6}$. After this the slope becomes steeper and after walking up another 10 m, the man reaches the top of the tower. Distance of 'A' from the foot of the tower is

(a) $5(1+\sqrt{3})$ m (b) $\frac{5}{2}(1+\sqrt{3})$ m

(c) $5(\sqrt{3}-1)$ m (d) $\frac{5}{2}(\sqrt{3}-1)$ m

Sir, I am unable to solve the above problem. Please give me detailed solution of it. [Vinay, Kanpur]

- Dear Vinay, here I am giving the detailed solution of your problem.

Let PQ be the tower of height 'h' and A, B are the points of observations.

Then, $\angle QAP=\frac{\pi}{4}$, $\angle BAP=\frac{\pi}{6}$, $AB=10$ m and $BQ=10$ m.

Now, $\angle QAB=\angle AQB$ [$\because$ angles opposite to equal sides are equal]

$$\angle ABQ=\pi-\frac{\pi}{6}=\frac{5\pi}{6}$$

Applying cosine rule in ΔABQ, we get

$$AQ^2=AB^2+BQ^2-2AB\cdot BQ\cdot\cos\frac{5\pi}{6}$$

$$=100+100+200\cdot\frac{\sqrt{3}}{2}$$

$$=200+100\sqrt{3}=100(2+\sqrt{3})$$

$$\Rightarrow AQ=10\sqrt{2+\sqrt{3}}$$

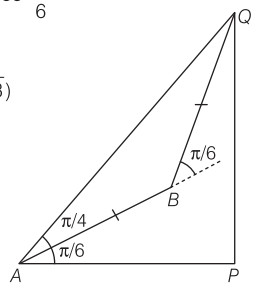
$$AP=AQ\cdot\cos\frac{\pi}{4}$$

$$=\frac{10\sqrt{2+\sqrt{3}}}{\sqrt{2}}$$

$$=5\sqrt{4+2\sqrt{3}}$$

$$=5\sqrt{(\sqrt{3}+1)^2}$$

$$AP=5(1+\sqrt{3})\text{ m}$$



USE OF COMPLEX NUMBERS IN COORDINATE GEOMETRY

Coverage of special topics for JEE Advanced

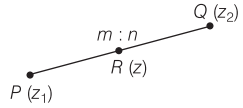
In this topic, we will learn the use of complex numbers in coordinate geometry. Here, we will see equation of various curves such as circle, hyperbola, ellipse in complex form. In JEE advanced, atleast one question is asked from it.

Section Formula

Let z_1 and z_2 be the affixes of two points P and Q respectively in the argand plane. Then, the affix of a point R dividing PQ

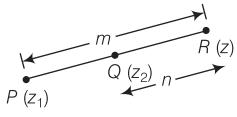
(i) internally in the ratio $m:n$ is

$$z = \frac{mz_2 + nz_1}{m+n}$$



(ii) externally in the ratio $m:n$ is

$$z = \frac{mz_2 - nz_1}{m-n}$$



KEY RESULTS

(i) If $R(z)$ is the mid-point of PQ, then affix of R is $\frac{z_1 + z_2}{2}$.

(ii) If z_1, z_2, z_3 are affixes of the vertices of a triangle, then the affix of its centroid is $\frac{z_1 + z_2 + z_3}{3}$.

Example 1 The points z_1, z_2, z_3, z_4 in the complex plane are the vertices of a parallelogram taken in order if and only if [IIT JEE 1983]

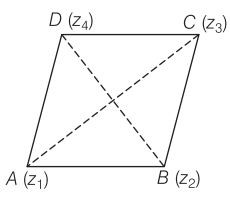
(a) $z_1 + z_4 = z_2 + z_3$ (b) $z_1 + z_3 = z_2 + z_4$
(c) $z_1 + z_2 = z_3 + z_4$ (d) None of these

Sol. ***(b)*** Since, z_1, z_2, z_3, z_4 are the vertices of parallelogram.

$\therefore$ Mid-point of AC = Mid-point of BD

$$\Rightarrow \quad \frac{z_1 + z_3}{2} = \frac{z_2 + z_4}{2}$$

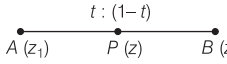
$$\Rightarrow \quad z_1 + z_3 = z_2 + z_4$$



Example 2 Let z_1 and z_2 be two distinct complex numbers and $z = (1-t)z_1 + tz_2$ for some real number t with $0 < t < 1$. If arg (w) denotes the principal argument of a non-zero complex number w, then [IIT JEE 2010]

(a) $|z - z_1| + |z - z_2| = |z_1 - z_2|$
(b) $\arg(z - z_1) = \arg(z - z_2)$
(c) $\begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$
(d) $\arg(z - z_1) = \arg(z_2 - z_1)$

Sol. ***(a, c, d)*** Given, $z = \frac{(1-t)z_1 + tz_2}{(1-t) + t}$



Clearly, z divides z_1 and z_2 in the ratio of $t : (1-t)$, $0 < t < 1$.

$\therefore \quad AP + BP = AB$

$\Rightarrow \quad |z - z_1| + |z - z_2| = |z_1 - z_2|$

and $\quad \arg(z - z_1) = \arg(z_2 - z) = \arg(z_2 - z_1)$

Also, $\quad \arg(z - z_1) = \arg(z_2 - z_1)$

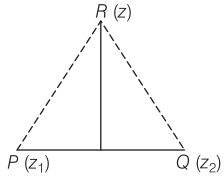
$$\Rightarrow \quad \arg\left(\frac{z - z_1}{z_2 - z_1}\right) = 0$$

$\therefore \quad \frac{z - z_1}{z_2 - z_1}$ is purely real.

$$\Rightarrow \quad \frac{z - z_1}{z_2 - z_1} = \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1} \text{ or } \begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$$

Equation of Perpendicular Bisector

If $P(z_1)$ and $Q(z_2)$ are two fixed points and $R(z)$ is any point on the perpendicular bisector of PQ, i.e. a point equidistant point from P and Q, then



The equation of perpendicular bisector is given as

$$|z - z_1| = |z - z_2|$$

Further $|z - z_1|^2 = |z - z_2|^2$

$\Rightarrow \quad (z - z_1)(\bar{z} - \bar{z}_1) = (z - z_2)(\bar{z} - \bar{z}_2)$

$\Rightarrow \quad z(\bar{z}_1 - \bar{z}_2) + \bar{z}(z_1 - z_2) = |z_1|^2 - |z_2|^2$

which is also another form of equation of perpendicular bisector.

REMEMBER

(i) Distance between two points $P(z_1)$ and $Q(z_2)$ is $PQ = |z_2 - z_1|$.

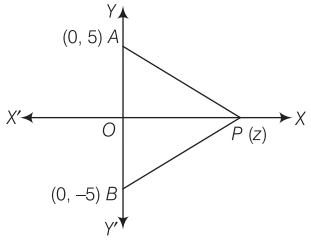
(ii) $|z|$ denotes the distance of point $P(z)$ from origin.

Example 3 The complex number $z = x + iy$ which satisfies the equation $\left|\frac{z - 5i}{z + 5i}\right| = 1$ lies on [IIT JEE 1981]

(a) the X-axis
(b) the straight line $y = 5$
(c) the circle passing through the origin
(d) None of the above

Sol. ***(a)*** We have, $\left|\frac{z - 5i}{z + 5i}\right| = 1 \Rightarrow |z - 5i| = |z + 5i|$

$\Rightarrow \quad |z - 5i| = |z - (-5i)| \Rightarrow PA = PB$



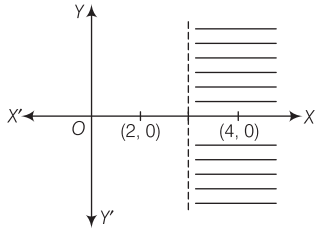
$\therefore$ P lies on the perpendicular bisector of AB, which is X-axis.

Example 4 The inequality $|z - 4| < |z - 2|$ represents the region given by [IIT JEE 1982]

(a) Re $(z) \geq 0$ *(b)* Re $(z) < 0$
(c) Re $(z) > 0$ *(d)* None of these

Sol. ***(d)*** Given, $|z - 4| < |z - 2|$

Since, $|z - z_1| > |z - z_2|$ represents the region on right side of perpendicular bisector of z_1 and z_2.



$\therefore \quad |z - 2| > |z - 4| \Rightarrow$ Re $(z) > 3$ and Im $(z) \in R$

Example 5 If $z = x + iy$ and $w = (1 - iz)/(z - i)$, then $|w| = 1$ implies that, in the complex plane [IIT JEE 1983]

(a) z lies on the imaginary axis
(b) z lies on the real axis
(c) z lies on the unit circle
(d) None of the above

Sol. ***(b)*** Since, $|w| = 1$

$\Rightarrow \quad \left|\frac{1 - iz}{z - i}\right| = 1 \Rightarrow |z - i| = |1 - iz|$

$\Rightarrow \quad |z - i| = |z + i|$

$[\because |1 - iz| = |-i||z + i| = |z + i|]$

which is a perpendicular bisector of $(0, 1)$ and $(0, -1)$, i.e. X-axis.
Thus, z lies on real axis.

Example 6 Let z and w be two complex numbers such that $|z| \leq 1, |w| \leq 1$ and $|z + iw| = |z - i\bar{w}| = 2$, then z equals [IIT JEE 1995]

(a) 1 or i *(b)* i or $-i$ *(c)* 1 or -1 *(d)* i or -1

Sol. ***(c)*** Given, $|z + iw| = |z - i\bar{w}| = 2$

$\Rightarrow \quad |z - (-iw)| = |z - i\bar{w}| = 2$

$\Rightarrow \quad |z - (-iw) = |z - (-i\bar{w})|$

So, z lies on the perpendicular bisector of the line joining $-iw$ and $-i\bar{w}$.

Since, $-i\bar{w}$ is the mirror image of $-iw$ in the X-axis, so the locus of z is the X-axis.

Let $z = x + iy$ and $y = 0$

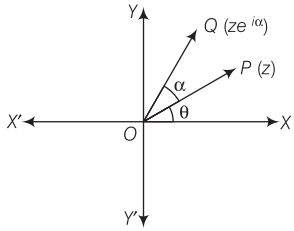
Now, $|z| \leq 1 \Rightarrow x^2 + 0^2 \leq 1 \Rightarrow -1 \leq x \leq 1$

$\therefore$ z may take values given in option (c).

Concept of Rotation

Let $z = re^{i\theta}$ be a complex number represented by a point P in the argand plane.

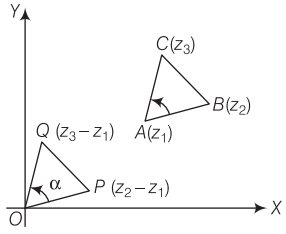
Then, $OP = r$ and $\angle XOP = \theta$



Now, $ze^{i\alpha} = re^{i\theta} \cdot e^{i\alpha} = re^{i(\theta + \alpha)}$

In other words, to obtain the point representing $ze^{i\alpha}$, we rotate **OP** through angle α in anti-clockwise direction.

In general, let z_1, z_2 and z_3 be the vertices of a ΔABC described in anti-clockwise sense. Draw OP and OQ parallel and equal to AB and AC respectively. Then, point P is $z_2 - z_1$ and Q is $z_3 - z_1$. If OP is rotated through an angle α in anti-clockwise sense and it coincides with OQ, then



$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{|z_3 - z_1|}{|z_2 - z_1|} e^{i\alpha}$$

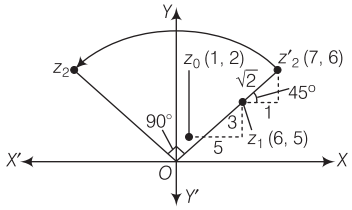
and if OP is rotated through an angle α in clockwise direction, then

$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{|z_3 - z_1|}{|z_2 - z_1|} e^{-i\alpha}$$

Example 7 A particle P starts from the point $z_0 = 1 + 2i$, where $i = \sqrt{-1}$. It moves first horizontally away from origin by 5 units and then vertically away from origin by 3 units to reach a point z_1. From z_1, the particle moves $\sqrt{2}$ units in the direction of the vector $\hat{i} + \hat{j}$ and then it moves through an angle $\pi/2$ in anti-clockwise direction on a circle with centre at origin to reach a point z_2. The point z_2 is given by **[IIT JEE 2008]**

(a) $6 + 7i$
(b) $-7 + 6i$
(c) $7 + 6i$
(d) $-6 + 7i$

Sol. **(d)** From $z_0(1, 2)$ when particle moves 5 units horizontally and 3 units vertically, it reaches the position $z_1(6, 5)$.



Now, $z_2' = (6 + \sqrt{2}\cos 45°, 5 + \sqrt{2}\sin 45°)$
$= (7, 6)$
$= 7 + 6i$

By rotation about (0, 0),

$$\frac{z_2}{z_2'} = e^{i\pi/2}$$

$$\Rightarrow \quad z_2 = z_2' e^{i\pi/2}$$

$$= (7 + 6i)\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$$= (7 + 6i)(i)$$

$$= 7i + 6i^2$$

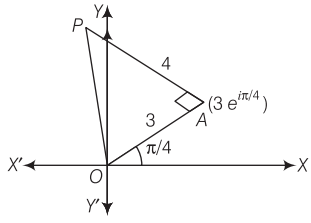
$$= -6 + 7i$$

Example 8 A man walks a distance of 3 units from origin towards the North-East (N 45° E) directions. From there, he walks a distance of 4 units towards the North-West (N 45° W) direction to reach a point P. Then, the position of P in the argand plane is **[IIT JEE 2007]**

(a) $3e^{i\pi/4} + 4i$ (b) $(3 - 4i)e^{i\pi/4}$
(c) $(4 + 3i)e^{i\pi/4}$ (d) $(3 + 4i)e^{i\pi/4}$

Sol. **(d)** Let $OA = 3$, so that the complex number associated with A is $3e^{i\pi/4}$. If z is the complex number associated with P, then

$$\frac{z - 3e^{i\pi/4}}{0 - 3e^{i\pi/4}} = \frac{4}{3}e^{-i\pi/2} = -\frac{4i}{3}$$



$$\Rightarrow \quad 3z - 9e^{i\pi/4} = 12ie^{i\pi/4}$$

$$\Rightarrow \quad z = (3 + 4i)e^{i\pi/4}$$

Example 9 $ABCD$ is a rhombus. Its diagonals AC and BD intersect at the point M and satisfy $BD = 2AC$. If the points D and M represent the complex numbers $1 + i$ and $2 - i$ respectively, then A represents the complex numbers ....or... . **[IIT JEE 1993]**

Sol. Since, M is the centre of rhombus.

$\therefore$ By rotating D about M through an angle of $\pm\pi/2$, we get possible position of A.

$$\Rightarrow \quad \frac{z_3 - z_2}{z_1 - z_2} = \frac{|z_3 - z_2|}{|z_1 - z_2|} e^{\pm i\pi/2}$$

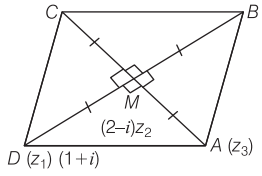
$$\Rightarrow \quad \frac{z_3 - (2 - i)}{-1 + 2i} = \frac{1}{2}(\pm i)$$

$$\Rightarrow \quad z_3 = (2 - i) \pm \frac{1}{2} i\,(2i - 1)$$

$$= 2 - i \pm \frac{1}{2}(-2 - i)$$

$$= \frac{4 - 2i - 2 - i}{2}, \frac{4 - 2i + 2 + i}{2}$$

$$= 1 - \frac{3}{2}i,\ 3 - \frac{i}{2}$$



$\therefore$ A is either $\left(1 - \frac{3i}{2}\right)$ or $\left(3 - \frac{i}{2}\right)$.

Some Important Results on Triangle

(i) Area of the triangle with vertices z_1, z_2 and z_3 is
$$\Delta = \frac{1}{4}\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} \text{ sq unit.}$$

(ii) The triangle whose vertices are the points z_1, z_2, z_3 on the argand plane is an equilateral triangle iff
$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$
or
$$\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$$

(iii) If the complex numbers z_1, z_2, z_3 are the vertices of an equilateral triangle and if z_0 is the circumcentre of the triangle, then $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$.

(iv) If z_1, z_2, z_3 are the vertices of an isosceles triangle, then $|z_3 - z_2| = |z_2 - z_1|$.

(v) If z_1, z_2, z_3 are the vertices of a right angled triangle, then $\left[\frac{z_3 - z_2}{z_2 - z_1}\right] = \pm\frac{\pi}{2}$.

REMEMBER

In an equilateral triangle, centroid, orthocentre and circumcentre coincides.

Example 10 The complex numbers z_1, z_2 and z_3 satisfying $\frac{z_1 - z_3}{z_2 - z_3} = \frac{1 - i\sqrt{3}}{2}$ are the vertices of a triangle which is [IIT JEE 2001]

(a) of area zero (b) right angled isosceles
(c) equilateral (d) obtuse angled isosceles

Sol. **(c)** Given,
$$\frac{z_1 - z_3}{z_2 - z_3} = \frac{1 - i\sqrt{3}}{2}$$
$$\Rightarrow \frac{z_2 - z_3}{z_1 - z_3} = \frac{2}{1 - i\sqrt{3}} = \frac{1 + i\sqrt{3}}{2} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$$
$$\Rightarrow \arg\left(\frac{z_2 - z_3}{z_1 - z_3}\right) = \frac{\pi}{3} \text{ and also } \left|\frac{z_2 - z_3}{z_1 - z_3}\right| = 1$$
Therefore, triangle is equilateral.

Example 11 Let the complex numbers z_1, z_2 and z_3 be the vertices of an equilateral triangle. Let z_0 be the circumcentre of the triangle . Then, prove that $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$. [IIT JEE 1981]

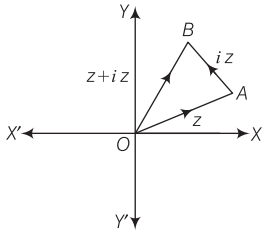
Sol. Since, z_1, z_2, z_3 are the vertices of an equilateral triangle.
$$\therefore \text{Circumcentre } (z_0) = \text{Centroid}\left(\frac{z_1 + z_2 + z_3}{3}\right) \quad \ldots(i)$$
Also, for equilateral triangle,
$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1 \quad \ldots(ii)$$
On squaring Eq. (i), we get
$$9z_0^2 = z_1^2 + z_2^2 + z_3^2 + 2(z_1z_2 + z_2z_3 + z_3z_1)$$
$$\Rightarrow 9z_0^2 = z_1^2 + z_2^2 + z_3^2 + 2(z_1^2 + z_2^2 + z_3^2)$$
$$\Rightarrow 3z_0^2 = z_1^2 + z_2^2 + z_3^2$$

Example 12 Prove that the complex numbers z_1, z_2 and the origin form an equilateral triangle only, if $z_1^2 + z_2^2 - z_1z_2 = 0$. [IIT JEE 1983]

Sol. Since, z_1, z_2 and origin form an equilateral triangle.
$$\Rightarrow z_1^2 + z_2^2 + 0^2 = z_1z_2 + z_2\cdot 0 + 0\cdot z_1$$
$$\Rightarrow z_1^2 + z_2^2 = z_1z_2$$
$$\Rightarrow z_1^2 + z_2^2 - z_1z_2 = 0$$
Hence proved.

Example 13 Show that the area of the triangle on the argand diagram formed by the complex number z, iz and $z + iz$ is $\frac{1}{2}|z|^2$. [IIT JEE 1986]

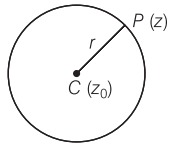
Sol. We have, $iz = ze^{i\pi/2}$. This implies that iz is the vector obtained by rotating vector $\mathbf{z}$ in anti-clockwise direction through 90°. Therefore, $OA \perp AB$.



$$\therefore \text{Area of } \Delta OAB = \frac{1}{2}\times OA \times OB$$
$$= \frac{1}{2}|z||iz| = \frac{1}{2}|z|^2$$

Equation of Circle

Let $C(z_0)$ be the centre of the circle with radius r, then the equation of circle is given as



$$|z - z_0| = r$$
$$\Rightarrow |z - z_0|^2 = r^2$$
$$\Rightarrow (z - z_0)(\bar{z} - \bar{z}_0) = r^2$$
$$\Rightarrow z\bar{z} - z_0\bar{z} - z\bar{z}_0 + z_0\bar{z}_0 = r^2$$
$$\Rightarrow z\bar{z} + (-z_0)\bar{z} + (-\bar{z}_0)z + |z_0|^2 - r^2 = 0$$
$$\Rightarrow z\bar{z} + a\bar{z} + \bar{a}z + b = 0$$
The above equation is general equation of circle whose centre is at '$-a$' and radius is $\sqrt{|a|^2 - b}$.

KEY POINTS

(i) If $|a|^2 > b$, then circle is real.

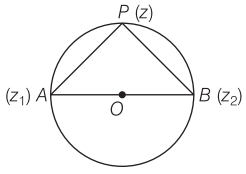
(ii) If $|a|^2 = b$, then circle is point circle.

(iii) If $|a|^2 < b$, then circle is imaginary.

Equation of Circle in Diametric Form

The equation of the circle having z_1 and z_2 as the end points of a diameter is

$$(z - z_1)(\bar{z} - \bar{z}_2) + (z - \bar{z}_1)(z - z_2) = 0$$



If z_1 and z_2 are fixed complex numbers, then the locus of a point z satisfying $\arg\left(\frac{z - z_1}{z - z_2}\right) = \pm\frac{\pi}{2}$ is a circle having z_1 and z_2 as the end points of a diameter.

Example 14 For all complex numbers z_1, z_2 satisfying $|z_1| = 12$ and $|z_2 - 3 - 4i| = 5$, the minimum value of $|z_1 - z_2|$ is [IIT JEE 2002]

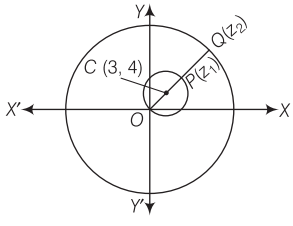
(a) 0 (b) 2 (c) 7 (d) 17

Sol. **(b)** Clearly, $|z_1| = 12$

$\Rightarrow z_1$ is a variable point on the circle $|z| = 12$.

and $|z_2 - (3 + 4i)| = 5 \Rightarrow z_2$ is a variable point on the circle $|z - (3 + 4i)| = 5$

Circles $|z| = 12$ and $|z - (3 + 4i)| = 5$ are drawn as shown in figure below



It is evident from above figure that z_1 and z_2 are the affixes of points P and Q respectively

and $\min|z_1 - z_2| = PQ = OQ - OP = 12 - 10 = 2$

Example 15 If z is a complex number such that $|z| \geq 2$, then the minimum value of $\left|z + \frac{1}{2}\right|$ [IIT JEE 2014]

(a) is equal to 5/2

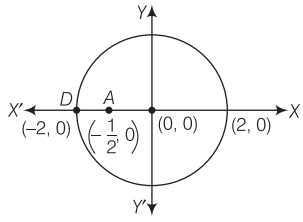
(b) lies in the interval (1, 2)

(c) is strictly greater than 5/2

(d) is strictly greater than 3/2 but less than 5/2

Sol. **(b)** $|z| \geq 2$ is the region on or outside circle whose centre is (0, 0) and radius is 2.

$\min\left|z + \frac{1}{2}\right|$ is distance of z, which lies on circle $|z| = 2$ from $\left(-\frac{1}{2}, 0\right)$.



$\therefore \min\left|z + \frac{1}{2}\right|$ = Distance of $\left(-\frac{1}{2}, 0\right)$ from (–2,0)

$$= \sqrt{\left(-2 + \frac{1}{2}\right)^2 + 0} = \frac{3}{2}$$

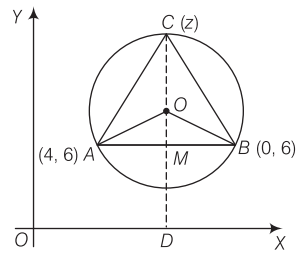
Example 16 Let $z_1 = 10 + 6i$ and $z_2 = 4 + 6i$. If z is any complex number such that the argument of $(z - z_1)/(z - z_2)$ is $\frac{\pi}{4}$, then prove that $|z - 7 - 9i| = 3\sqrt{2}$. [IIT JEE 1991]

Sol. Since, $z_1 = 10 + 6i$, $z_2 = 4 + 6i$

$\arg\left(\frac{z - z_1}{z - z_2}\right) = \frac{\pi}{4}$ will represent locus of z which is a circle.

$\because$ $\angle ACB = 45°$

$\angle AOB = 90°$



In ΔAOB, $r = 3\sqrt{2}$

and in ΔAOM,

$$OM = \sqrt{(3\sqrt{2})^2 - 3^2} = \sqrt{9} = 3$$

$\therefore$ $OD = OM + DM = 3 + 6 = 9$

Hence, centre of the circle is (7, 9) and radius $= 3\sqrt{2}$

$\therefore$ $|z - (7 + 9i)| = 3\sqrt{2}$ **Hence proved.**

Example 17 Find the centre and radius of the circle formed by all the points represented by $z = x + iy$ satisfying the relation $\left|\frac{z - \alpha}{z - \beta}\right| = k\ (k \neq 1)$, where α and β are constant complex numbers given by $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$. [IIT JEE 2004]

Sol. We know that, $|z|^2 = z \cdot \bar{z}$

Given, $\dfrac{|z-\alpha|^2}{|z-\beta|^2} = k^2$

$\Rightarrow (z-\alpha)(\bar{z}-\bar{\alpha}) = k^2 (z-\beta)(\bar{z}-\bar{\beta})$

$\Rightarrow |z|^2 - \alpha\bar{z} - \bar{\alpha} z + |\alpha|^2 = k^2(|z|^2 - \beta\bar{z} - \bar{\beta} z + |\beta|^2)$

$\Rightarrow |z|^2(1-k^2) - (\alpha - k^2\beta)\bar{z} - (\bar{\alpha} - \bar{\beta}k^2)z + (|\alpha|^2 - k^2|\beta|^2) = 0$

$\Rightarrow |z|^2 - \dfrac{\alpha - k^2\beta}{(1-k^2)}\bar{z} - \dfrac{\bar{\alpha} - \bar{\beta}k^2}{(1-k^2)}z + \dfrac{|\alpha|^2 - k^2|\beta|^2}{(1-k^2)} = 0$

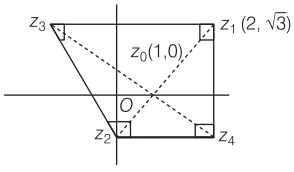
$\therefore$ Centre $= \dfrac{\alpha - k^2\beta}{1-k^2}$

Radius $= \sqrt{\left(\dfrac{\alpha - k^2\beta}{1-k^2}\right)\left(\dfrac{\bar{\alpha} - k^2\bar{\beta}}{1-k^2}\right) - \left(\dfrac{\alpha\bar{\alpha} - k^2\beta\bar{\beta}}{1-k^2}\right)}$

$= \left|\dfrac{k(\alpha-\beta)}{1-k^2}\right|$

Example 18 If one of the vertices of the square circumscribing the circle $|z-1| = \sqrt{2}$ is $2 + \sqrt{3}\,i$. Then, find the other vertices of square. **[IIT JEE 2005]**

Sol. Here, centre of circle is (1, 0), is also the mid-point of diagonals of square.



$\Rightarrow \dfrac{z_1 + z_2}{2} = z_0$

$\Rightarrow z_2 = -\sqrt{3}\,i$ $\quad [\because z_0 = 1 + 0i]$

and $\dfrac{z_3 - 1}{z_1 - 1} = e^{\pm i\pi/2}$

$\Rightarrow z_3 = 1 + (1 + \sqrt{3}i)\cdot(\pm i)$

$= 1 \pm i(1 + \sqrt{3}i) = (1 \mp \sqrt{3}) \pm i$

$\therefore z_3 = (1 - \sqrt{3}) + i$ and $z_4 = (1 + \sqrt{3}) - i$

Example 19 If the complex numbers z_1, z_2 and z_3 represent the vertices of an equilateral triangle such that $|z_1| = |z_2| = |z_3|$, then $z_1 + z_2 + z_3 = 0$. (True/False) **[IIT JEE 1984]**

Sol. True

Since, z_1, z_2, z_3 are the vertices of equilateral triangle and $|z_1| = |z_2| = |z_3|$.

$\Rightarrow z_1, z_2, z_3$ lie on a circle with centre at origin.

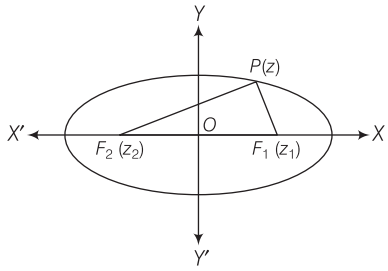
$\Rightarrow$ Circumcentre = Centroid

$\Rightarrow 0 = \dfrac{z_1 + z_2 + z_3}{3}$

$\therefore z_1 + z_2 + z_3 = 0$

Ellipse

If $F_1(z_1)$ and $F_2(z_2)$ are two fixed points and $P(z)$ is a variable point, such that



$|z - z_1| + |z - z_2| = k = \text{constant} > |z_1 - z_2|$

$\Rightarrow PF_1 + PF_2 = k$

Thus, the locus of z represents an ellipse.

Eccentricity of ellipse $= \dfrac{F_1F_2}{PF_1 + PF_2}$

$= \dfrac{|z_1 - z_2|}{|z - z_1| + |z - z_2|}$

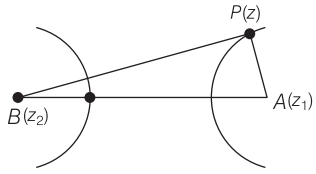
Hyperbola

If $A(z_1)$ and $B(z_2)$ are two fixed points and $P(z)$ is a variable point such that

$||z - z_1| - |z - z_2|| = k = \text{constant} < |z_1 - z_2|$

$\Rightarrow |AP - BP| = k$

Thus, the locus of point $P(z)$ is a hyperbola.



Eccentricity $= \dfrac{AB}{|AP - BP|}$

$= \dfrac{|z_1 - z_2|}{||z - z_1| - |z - z_2||}$

Example 20 The set of point z satisfying $|z + 4| + |z - 4| = 10$ is **[IIT JEE 2010]**

(a) straight line (b) circle

(c) ellipse (d) hyperbola

Sol. **(c)** Sum of distance of z from (4, 0) and (– 4, 0) is a constant 10, hence locus of z is ellipse with semi-major axis 5 and focus at (± 4, 0), $ae = 4$

$\Rightarrow e = \dfrac{4}{5}$

Smart Practice

Single Option Correct Type

1. If z_1, z_2, z_3 are three complex numbers such that $5z_1 - 13z_2 + 8z_3 = 0$, then

(a) z_1, z_2, z_3 are collinear (b) z_1, z_2, z_3 form a triangle
(c) z_1, z_2, z_3 are concyclic (d) None of these

2. Intercept made by the circle $z\bar{z} + \bar{\alpha}z + \alpha\bar{z} + r = 0$ on the real axis on complex plane is

(a) $\sqrt{(\alpha + \bar{\alpha})^2 - r}$ (b) $\sqrt{(\alpha + \bar{\alpha})^2 - 2r}$
(c) $\sqrt{(\alpha + \bar{\alpha})^2 - 3r}$ (d) $\sqrt{(\alpha + \bar{\alpha})^2 - 4r}$

3. If $A(z_1), B(z_2), C(z_3)$ are the vertices of an equilateral ΔABC, then $\arg\left(\frac{2z_1 - z_2 - z_3}{z_3 - z_2}\right)$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{6}$

4. The radius of the circle $\left|\frac{z - i}{z + i}\right| = 3$ is

(a) $\frac{5}{4}$ (b) $\frac{3}{4}$
(c) $\frac{1}{4}$ (d) None of these

5. For all complex numbers z, if $|z + 3i| + |z - i| = 8$, then the locus of z is

(a) an ellipse of eccentricity 1/2 and major axis along X-axis
(b) an ellipse of eccentricity 1/2 and major axis along Y-axis
(c) an ellipse of eccentricity $\frac{1}{\sqrt{2}}$ and major axis along Y-axis.
(d) None of the above

One or More than One Options Correct Type

6. Locus of complex number satisfying $[(z - 5 + 4i)/(z + 3 - 2i)] = -\pi/4$ is the arc of a circle

(a) whose radius is $5\sqrt{2}$
(b) whose radius is 5
(c) whose length (of arc) is $\frac{15\pi}{\sqrt{2}}$
(d) whose centre is $-2 - 5i$

7. If $|(z - z_1)/(z - z_2)| = 3$, where z_1 and z_2 are fixed complex numbers and z is a variable complex number, then z lies on a

(a) circle with z_1 as its interior point
(b) circle with z_2 as its interior point
(c) circle with z_1 as its exterior point
(d) circle with z_2 as its exterior point

8. $P(z_1), Q(z_2), R(z_3)$ and $S(z_4)$ are four complex numbers representing the vertices of a rhombus taken in order on the complex plane, then which one of the following is/are correct?

(a) $\frac{z_1 - z_4}{z_2 - z_3}$ is purely real
(b) $\text{amp}\left(\frac{z_1 - z_4}{z_2 - z_4}\right) = \text{amp}\left(\frac{z_2 - z_4}{z_3 - z_4}\right)$
(c) $\frac{z_1 - z_3}{z_2 - z_4}$ is purely imaginary
(d) It is not necessary that $|z_1 - z_3| \neq |z_2 - z_4|$

Linked Comprehension Type

Passage (Q. Nos. 9-11) Consider the equation,

$$\bar{z} = \bar{a} + \frac{r^2}{z - a}$$

9. The locus of z is

(a) straight line (b) circle
(c) ellipse (d) hyperbola

10. The centre of the circle is

(a) a (b) $-a$ (c) $\bar{a}$ (d) $-\bar{a}$

11. Radius of the circle is

(a) $\frac{r}{2}$ (b) $2r$ (c) r (d) $\frac{3r}{2}$

Single Integer Answer Correct Type

12. The minimum value of $(k - 20)$ for which the equation $z\bar{z} + (-3 + 4i)\bar{z} + (-3 - 4i)z + k = 0$ represents a circle, is _____.

13. If z is any complex number satisfying $|z - 3 - 2i| \leq 2$, then the minimum value of $|2z - 6 + 5i|$ is _____.

14. If the equation $|z - a| + |z - b| = 3$ represents an ellipse and $a, b \in C$, where a is fixed, then the locus b is a point lies inside a circle of radius _____.

Match the Column Type

15. Which of the values in Column II are satisfied by the conditions given in Column I?

Column I	Column II
A. If $\|z - 2i\| + \|z - 7i\| = k$, then locus of z is an ellipse, if k is equal to	p. 7
B. If $\|(2z - 3)/(3z - 2)\| = k$, then locus of z is a circle, if 2/3 is a point inside circle and 3/2 is outside the circle, if k is equal to	q. 8
C. If $\|z - 3\| - \|z - 4i\| = k$, then locus of z is a hyperbola, if k is equal to	r. 2
D. If $\|z - (3 + 4i)\| = (k/50)\|a\bar{z} + \bar{a}z + b\|$, where $a = 3 + 4i$, then locus of z is a hyperbola with k is equal to	s. 4
	t. 5

Answers with Explanation

1. ***(a)*** We have,

$5z_1 - 13z_2 + 8z_3 = 0$

$\Rightarrow \quad 13z_2 = 5z_1 + 8z_3$

$\Rightarrow \quad z_2 = \dfrac{5z_1 + 8z_3}{13}$

$\Rightarrow \quad z_2 = \dfrac{5z_1 + 8z_3}{5 + 8}$

Hence, z_1, z_2, z_3 are collinear.

2. ***(d)*** Point, where the circle cuts the X-axis, is $z = \bar{z}$.

Hence, substituting $z = \bar{z}$ in the equation of circle, we get

$z^2 + \bar{\alpha} z + \alpha z + r = 0$

or $\quad z^2 + (\alpha + \bar{\alpha})\, z + r = 0$

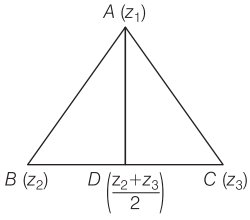
$\Rightarrow \quad AB = |z_1 - z_2|$

where, A and B are points of intersection of circle with X-axis.

$\therefore \quad AB = \sqrt{(z_1 + z_2)^2 - 4z_1 z_2} = \sqrt{(\alpha + \bar{\alpha})^2 - 4r}$

3. ***(b)*** We have, $\arg\left(\dfrac{z_1 - \dfrac{z_2 + z_3}{2}}{z_3 - z_2}\right) = \dfrac{\pi}{2}$ $\quad [\because AD \perp BC]$

$\Rightarrow \quad \arg\left(\dfrac{2z_1 - z_2 - z_3}{z_3 - z_2}\right) = \dfrac{\pi}{2}$



4. ***(b)*** We have, $\left|\dfrac{z - i}{z + i}\right| = 3$

$\Rightarrow \quad |z - i| = 3|z + i|$

$\Rightarrow \quad |z - i|^2 = 9|z + i|^2$

$\Rightarrow \quad (z - i)(\bar{z} + i) = 9(z + i)(\bar{z} - i)$

$\Rightarrow \quad z\bar{z} - i\bar{z} + iz + 1 = 9(z\bar{z} - iz + i\bar{z} + 1)$

$\Rightarrow \quad 8z\bar{z} - 10iz + 10\, i\, \bar{z} + 8 = 0$

$\Rightarrow \quad z\bar{z} - \dfrac{5}{4} iz + \dfrac{5}{4} i\, \bar{z} + 1 = 0$

$\therefore \quad \text{Radius} = \sqrt{\left|\dfrac{5}{4} i\right|^2 - 1} = \dfrac{3}{4}$

5. ***(a)*** We have, $|z + 3i| + |z - i| = 8$

$\Rightarrow \quad PF_1 + PF_2 = 8$, where $F_1(0 - 3i)$, $F_2(0 + i)$

and $P(z)$ are points in the argand plane.

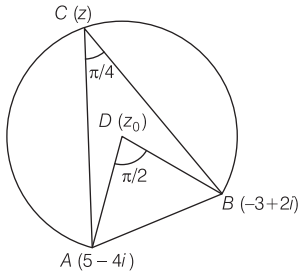
$\Rightarrow$ Locus of P is an ellipse having foci at F_1 and F_2 and major axis = 8

$\therefore \quad 2a = 8$ and $2ae = F_1F_2 = 4$

$\Rightarrow \quad a = 4$ and $ae = 2$

$\Rightarrow \quad e = \dfrac{1}{2}$

6. ***(a, c, d)*** $\because \dfrac{z_0 - (-3 + 2i)}{z_0 - (5 - 4i)} = i$



$\Rightarrow \quad z_0 + 3 - 2i = iz_0 - 5i - 4$

$\Rightarrow \quad z_0 = -2 - 5i$

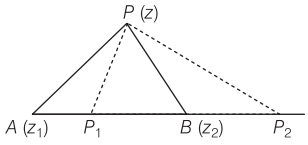
$\Rightarrow$ Radius $(AD) = |5 - 4i - (-2 - 5i)|$

$= |7 + i| = \sqrt{50} = 5\sqrt{2}$

$\therefore$ Length of arc $= \dfrac{3}{4}$ (Perimeter of circle)

$= \dfrac{3}{4}(2\pi \times 5\sqrt{2}) = \dfrac{15\pi}{\sqrt{2}}$

7. ***(b, c)*** Let internal and external bisectors of $\angle APB$ meet the line joining A and B at P_1 and P_2, respectively.



Hence, $AP_1 : P_1B = PA : PB \equiv 3 : 1$ (internal division)

$AP_2 : P_2B = PA : PB \equiv 3 : 1$ (external division)

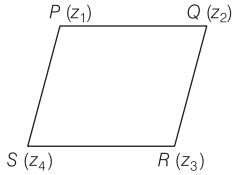
Thus, P_1 and P_2 are fixed points. Also, $\angle P_1PP_2 = \dfrac{\pi}{2}$

Thus, P lies on a circle having P_1P_2 as its diameter. Clearly, $B(z_2)$ lies inside this circle.

8. ***(a, b, c, d)***

(a) $PS \parallel QR \Rightarrow \arg\left(\dfrac{z_1 - z_4}{z_2 - z_3}\right) = 0$

$\Rightarrow \dfrac{z_1 - z_4}{z_2 - z_3}$ is purely real.



(b) Since, diagonal bisects the angle

$\Rightarrow \text{amp}\left(\dfrac{z_1 - z_4}{z_2 - z_4}\right) = \text{amp}\left(\dfrac{z_2 - z_4}{z_3 - z_4}\right)$

(c) Diagonals of rhombus are perpendicular. Hence, $(z_1 - z_3)/(z_2 - z_4)$ is purely imaginary.

(d) Diagonals of rhombus are not equal.

Hence, $|z_1 - z_3| \neq |z_2 - z_4|$.

Solutions (Q. Nos. 9-11)

We have, $\bar{z} = \bar{a} + \dfrac{r^2}{z-a} \Rightarrow \bar{z} - \bar{a} = \dfrac{r^2}{z-a}$

$\Rightarrow (z-a)(\bar{z}-\bar{a}) = r^2$ or $|z-a|^2 = r^2$ or $|z-a| = r$

Hence, locus of z is circle having centre a and radius r.

9. ***(b)*** **10.** ***(a)*** **11.** ***(c)***

12. ***(5)*** We have, $z\bar{z} + (-3+4i)\bar{z} + (-3-4i)z + k = 0$

This equation represents a circle with centre $= (3-4i)$

and radius $= \sqrt{|-3+4i|^2 - k} = \sqrt{25-k}$

For the circle to exist, we must have

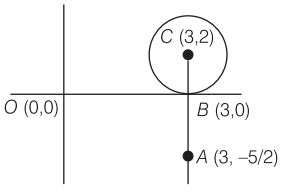
$25 - k \geq 0$

$\Rightarrow \quad k \leq 25$

$\therefore \quad k - 20 \leq 5$

13. ***(5)*** We have, $|z-3-2i| \leq 2$

$\Rightarrow z$ lies on or inside the circle having radius 2 and centre (3, 2).



$\therefore \quad \min|2z-6+5i| = \min\{2|z-3+(5/2)i|\}$

$= 2$ (minimum distance of any point on the circle to the point $(3, -5/2)$)

$= 2 \times \dfrac{5}{2} = 5$

14. ***(3)*** $|z-a| + |z-b| = 3$ represents an ellipse.

Now, $|a-b| < 3$

or $|b-a| < 3$

Hence, b lies inside the circle having centre a and radius 3.

15. A $\rightarrow$ p, q; B $\rightarrow$ p, q, r, s, t ; C $\rightarrow$ r, s; D $\rightarrow$ p,q

A. $|z-2i| + |z-7i| = k$ is ellipse, if $k > |7i-2i|$ or $k > 5$

B. $\left|\dfrac{2z-3}{3z-2}\right| = k \Rightarrow \left|\dfrac{z-\frac{3}{2}}{z-\frac{2}{3}}\right| = \dfrac{3k}{2}$

$\Rightarrow \quad \dfrac{3k}{2} > 1$

$\Rightarrow \quad k > \dfrac{2}{3}$

C. $|z-3| - |z-4i| = k$ is hyperbola, if $k < |3-4i| \Rightarrow 0 < k < 5$

D. $|z-(3+4i)| = \dfrac{k}{50}|a\bar{z} + \bar{a}z + b|$

$\Rightarrow |z-(3+4i)| = \dfrac{k}{5}\dfrac{|a\bar{z} + \bar{a}z + b|}{2|3+4i|}$

This is hyperbola, if $\dfrac{k}{5} > 1 \Rightarrow k > 5$

वस्तुनिष्ठ

मूलभूत शृंखला

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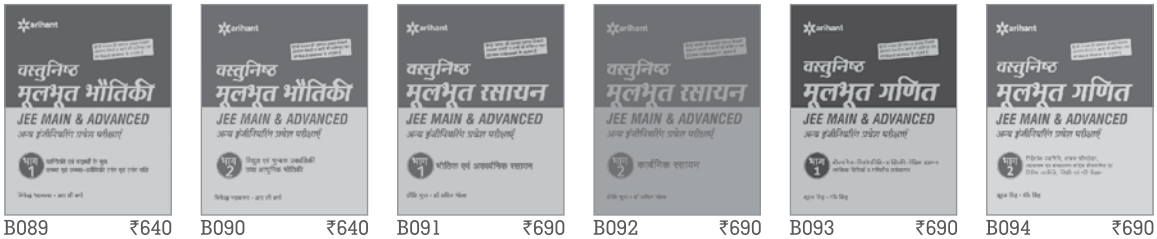
अन्य इंजीनियरिंग प्रवेश परीक्षा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हिन्दी माध्यम की एकमात्र पुस्तक जिसकी अध्ययन सामग्री व प्रश्नों का कठिनता स्तर JEE MAIN & ADVANCED के अनुरूप है

Love Aggarwal

EXEMPLAR SIMPLIFIED

EXTRACT OF EXTRA ORDINARY EXEMPLAR PROBLEM

1. If A and B are symmetric matrices of the same order, then $(AB' - BA')$ is a

(a) skew-symmetric matrix *(b)* null matrix
(c) symmetric matrix *(d)* None of these

2. Total number of possible matrices of order 3×3 with each entry 2 or 0 is

(a) 9 *(b)* 27
(c) 81 *(d)* 512

3. If A is a square matrix such that $A^2 = I$, then $(A - I)^3 + (A + I)^3 - 7A$ is equal to

(a) A *(b)* $I - A$
(c) $I + A$ *(d)* $3A$

4. Let $\Delta = \begin{vmatrix} Ax & x^2 & 1 \\ By & y^2 & 1 \\ Cz & z^2 & 1 \end{vmatrix}$ and $\Delta_1 = \begin{vmatrix} A & B & C \\ x & y & z \\ zy & zx & xy \end{vmatrix}$, then

(a) $\Delta_1 = -\Delta$ *(b)* $\Delta \neq \Delta_1$
(c) $\Delta - \Delta_1 = 0$ *(d)* None of these

5. If $x, y \in R$, then the determinant $\begin{vmatrix} \cos x & \sin x & 1 \\ \sin x & -\cos x & 1 \\ \cos(x+y) & \sin(x+y) & 0 \end{vmatrix}$ lies in the interval

(a) $[-\sqrt{2}, \sqrt{2}]$ *(b)* $[-1, 1]$
(c) $[-\sqrt{2}, 1]$ *(d)* $[1, \sqrt{2}]$

6. The determinant $\begin{vmatrix} b^2 - ab & b - c & bc - ac \\ ab - a^2 & a - b & b^2 - ab \\ bc - ac & c - a & ab - a^2 \end{vmatrix}$ is equal to

(a) $abc\,(b - c)(c - a)(a - b)$
(b) $(b - c)(c - a)(a - b)$
(c) $(a + b + c)(b - c)(c - a)(a - b)$
(d) None of the above

7. The number of distinct real roots of $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$ in the interval $\frac{-\pi}{4} \le x \le \frac{\pi}{4}$, is

(a) 0 *(b)* 2 *(c)* 1 *(d)* 3

8. If A, B and C are angles of a triangle, then the determinant $\begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$ is equal to

(a) 0 *(b)* -1
(c) 1 *(d)* None of these

9. The maximum value of $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 + \sin\theta & 1 \\ 1 + \cos\theta & 1 & 1 \end{vmatrix}$ is

(where, θ is real number)

(a) $\frac{1}{2}$ *(b)* $\frac{\sqrt{3}}{2}$ *(c)* $\sqrt{2}$ *(d)* $\frac{2\sqrt{3}}{4}$

10. If x, y and z are different from and $\begin{vmatrix} 1 + x & 1 & 1 \\ 1 & 1 + y & 1 \\ 1 & 1 & 1 + z \end{vmatrix} = 0$, then the value of $x^{-1} + y^{-1} + z^{-1}$ is

(a) xyz *(b)* $x^{-1}\, y^{-1}\, z^{-1}$
(c) $-x - y - z$ *(d)* -1

11. If the matrix $A = \begin{vmatrix} 0 & a & 3 \\ 2 & b & -1 \\ c & 1 & 0 \end{vmatrix}$ is skew-symmetric, then

(a) $a = -2, b = 0, c = -3$ *(b)* $a = 0, b = -3, c = -2$
(c) $a = -2, b = -3, c = 0$ *(d)* $a = 0, b = -2, c = -3$

12. $\begin{vmatrix} \text{cosec}^2\theta & \cot^2\theta & 1 \\ \cot^2\theta & \text{cosec}^2\theta & -1 \\ 42 & 40 & 2 \end{vmatrix}$ is equal to

(a) 1 *(b)* 0 *(c)* $\text{cosec}^2\theta$ *(d)* $\cot^2\theta$

13. In a ΔABC, if

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin C+\sin^2 C \end{vmatrix} = 0,$$

then ΔABC is

(a) right angled *(b)* isosceles
(c) equilateral *(d)* None of these

14. The value of $\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix}$ is

(a) a^3 *(b)* 1 *(c)* $(a-1)^3$ *(d)* 0

15. If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are vertices of an equilateral triangle whose each side is equal to a, then the value of $\begin{vmatrix} x_1 & y_1 & 2 \\ x_2 & y_2 & 2 \\ x_3 & y_3 & 2 \end{vmatrix}^2$ is

(a) a^4 *(b)* $3a^2$ *(c)* $(1/3)a^4$ *(d)* $3a^4$

16. If A is a 3×3 invertible matrix and k is any scalar, then $(kA)^{-1}$ is equal to

(a) $\frac{1}{k}A^{-1}$ *(b)* kA^{-1} *(c)* $\frac{1}{k}A$ *(d)* kA

17. If A is a symmetrix matrix, then A^3 is a

(a) skew-symmetric *(b)* symmetric
(c) identity *(d)* None of these

18. If the determinant $\begin{vmatrix} 3 & -2 & \sin 3\theta \\ -7 & 8 & \cos 2\theta \\ -11 & 14 & 2 \end{vmatrix} = 0$, then $\sin\theta$ is equal to

(a) 0 *(b)* $\frac{1}{2}$
(c) Both (a) and (b) *(d)* None of these

19. There are two values of a which makes determinant, $\Delta = \begin{vmatrix} 1 & -2 & 5 \\ 2 & a & 1 \\ 0 & 4 & 2a \end{vmatrix} = 28$, then sum of these numbers is

(a) 4 *(b)* 5 *(c)* – 4 *(d)* 9

20. If A and B are matrices of order 3 and $|A| = 5, |B| = 3$, then $|3AB|$ is equal to

(a) 45 *(b)* 105 *(c)* 90 *(d)* 405

Answers with Explanation

1. ***(a)*** Let $P = AB' - BA'$

$\therefore$ $P' = (AB' - BA')' = (AB')' - (BA')'$
$= (B')'A' - (A')'B' = BA' - AB'$
$= -(AB' - BA') = -P$

Hence, P is a skew-symmetric matrix.

2. ***(d)*** Total number of elements $= 3 \times 3 = 9$

$\therefore$ Total number of possible matrices with each entry 2 or 0 $= 2^9 = 512$

3. ***(a)*** We have, $(A - I)^3 + (A + I)^3 - 7A$

$= A^3 - 3A^2I + 3AI^2 - I^3 + A^3 + 3A^2I + 3AI^2 + I^3 - 7A$
$= 2A^3 + 6AI^2 - 7A$
$= 2(A^2)\cdot A + 6AI - 7A$ $[\because I^2 = I]$
$= 2A\cdot A + 6A - 7A$ $[\because AI = A]$
$= 2A^2 + 6A - 7A = 2A + 6A - 7A = A$ $[\because A^2 = A]$

4. ***(c)*** We have, $\Delta_1 = \begin{vmatrix} A & B & C \\ x & y & z \\ zy & zx & xy \end{vmatrix} = \begin{vmatrix} A & x & zy \\ B & y & zx \\ C & z & xy \end{vmatrix}$ $[\because |A| = |A'|]$

$= \frac{1}{xyz}\begin{vmatrix} Ax & x^2 & xyz \\ By & y^2 & xyz \\ Cz & z^2 & xyz \end{vmatrix} = \frac{xyz}{xyz}\begin{vmatrix} Ax & x^2 & 1 \\ By & y^2 & 1 \\ Cz & z^2 & 1 \end{vmatrix} = \begin{vmatrix} Ax & x^2 & 1 \\ By & y^2 & 1 \\ Cz & z^2 & 1 \end{vmatrix} = \Delta$

$\therefore$ $\Delta - \Delta_1 = 0$

5. ***(a)*** We have, $\Delta = \begin{vmatrix} \cos x & \sin x & 1 \\ \sin x & -\cos x & 1 \\ \cos(x+y) & \sin(x+y) & 0 \end{vmatrix}$

Applying $R_3 \to R_3 - \cos y\, R_1 + \sin y\, R_2$, we get

$$\Delta = \begin{vmatrix} \cos x & \sin x & 1 \\ \sin x & -\cos x & 1 \\ 0 & 0 & \sin y - \cos y \end{vmatrix}$$

Expanding along R_3, we get

$\Delta = (\sin y - \cos y)(\cos^2 x + \sin^2 x) = \sin y - \cos y$
$= \sqrt{2}\left[\frac{1}{\sqrt{2}}\sin y - \frac{1}{\sqrt{2}}\cos y\right]$
$= \sqrt{2}\left[\sin y\cos\frac{\pi}{4} - \cos y\sin\frac{\pi}{4}\right] = \sqrt{2}\sin\left(y - \frac{\pi}{4}\right)$

Now, $-1 \le \sin\left(y - \frac{\pi}{4}\right) \le 1$

$\Rightarrow$ $-\sqrt{2} \le \sqrt{2}\sin\left(y - \frac{\pi}{4}\right) \le \sqrt{2} \Rightarrow -\sqrt{2} \le \Delta \le \sqrt{2}$

$\therefore \Delta \in [-\sqrt{2}, \sqrt{2}]$

6. ***(d)*** Let $\Delta = \begin{vmatrix} b^2 - ab & b - c & bc - ac \\ ab - a^2 & a - b & b^2 - ab \\ bc - ac & c - a & ab - a^2 \end{vmatrix}$

$= \begin{vmatrix} b(b-a) & b-c & c(b-a) \\ a(b-a) & a-b & b(b-a) \\ c(b-a) & c-a & a(b-a) \end{vmatrix}$

$$= (b-a)^2 \begin{vmatrix} b & b-c & c \\ a & a-b & b \\ c & c-a & a \end{vmatrix}$$

[taking common $(b-a)$ from C_1 and C_3]

$$= (b-a)^2 \begin{vmatrix} b & 0 & c \\ a & 0 & b \\ c & 0 & a \end{vmatrix} \quad [\text{using } C_2 \to C_2 - (C_1 - C_3)]$$

$= 0$

7. ***(c)*** Given, $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$

Applying $R_1 \to R_1 + R_2 + R_3$, we get

$$\begin{vmatrix} \sin x + 2\cos x & \sin x + 2\cos x & \sin x + 2\cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$$

$$\Rightarrow (\sin x + 2\cos x) \begin{vmatrix} 1 & 1 & 1 \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$$

[taking common $(\sin x + 2\cos x)$ from R_1]

$$\Rightarrow (\sin x + 2\cos x) \begin{vmatrix} 1 & 0 & 0 \\ \cos x & \sin x - \cos x & 0 \\ \cos x & 0 & \sin x - \cos x \end{vmatrix} = 0$$

[using $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$]

$\Rightarrow (\sin x + 2\cos x)[1(\sin x - \cos x)^2] = 0$

$\Rightarrow \sin x + 2\cos x = 0$ or $(\sin x - \cos x)^2 = 0$

$\Rightarrow \tan x = -2$ or $\tan x = 1$

For $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$, $\tan x \neq -2$.

But $\tan x = 1 \Rightarrow x = \frac{\pi}{4}$

8. ***(a)*** Given, A, B and C are the angles of a triangle.

$\therefore A + B + C = \pi \Rightarrow A + B = \pi - C$

Now, $\cos(A+B) = \cos(\pi - C) = -\cos C$

$\Rightarrow \cos A \cos B - \sin A \sin B = -\cos C$

$\Rightarrow \cos A \cos B + \cos C = \sin A \sin B$ ...(i)

Similarly, $\cos A \cos C + \cos B = \sin A \sin C$ ...(ii)

and $\sin(B+C) = \sin(\pi - A) = \sin A$ ...(iii)

Now, let $\Delta = \begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$

Expanding along R_1, we get

$\Delta = -1(1 - \cos^2 A) + \cos C(\cos C + \cos A \cos B) + \cos B(\cos B + \cos A \cos C)$

$= -\sin^2 A + \cos C(\sin A \sin B) + \cos B(\sin A \sin C)$ [from Eqs. (i) and (ii)]

$= -\sin^2 A + \sin A(\sin B \cos C + \cos B \sin C)$

$= -\sin^2 A + \sin A \sin(B + C)$

$= -\sin^2 A + \sin^2 A$ [using Eq. (iii)]

$= 0$

9. ***(a)*** Given, $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1+\cos\theta & 1 & 1 \end{vmatrix}$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ 1 & \sin\theta & 0 \\ 1+\cos\theta & -\cos\theta & -\cos\theta \end{vmatrix}$$

$$= 1(-\sin\theta\cos\theta) = \frac{-1}{2}\sin 2\theta$$

For maximum value of Δ, $\sin 2\theta$ will be minimum.

$\therefore \Delta_{max} = -\frac{1}{2}(-1) = \frac{1}{2}$ [$\because$ minimum value of $\sin x$ is -1]

10. ***(d)*** We have, $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = 0$

Taking x, y and z common from C_1, C_2 and C_3 respectively, we get

$$xyz \begin{vmatrix} \frac{1}{x}+1 & \frac{1}{y} & \frac{1}{z} \\ \frac{1}{x} & \frac{1}{y}+1 & \frac{1}{z} \\ \frac{1}{x} & \frac{1}{y} & \frac{1}{z}+1 \end{vmatrix} = 0$$

$$\Rightarrow xyz \begin{vmatrix} 1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z} & \frac{1}{y} & \frac{1}{z} \\ 1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z} & 1+\frac{1}{y} & \frac{1}{z} \\ 1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z} & \frac{1}{y} & 1+\frac{1}{z} \end{vmatrix} = 0$$

[applying $C_1 \to C_1 + C_2 + C_3$]

$$\Rightarrow (xyz)\left(1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \begin{vmatrix} 1 & 1/y & 1/z \\ 1 & 1+1/y & 1/z \\ 1 & 1/y & 1+1/z \end{vmatrix} = 0$$

[taking common $\left(1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$ from C_1]

$$\Rightarrow xyz\left(1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \begin{vmatrix} 1 & 1/y & 1/z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

[applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$]

$$\Rightarrow xyz\left(1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)1(1-0) = 0$$

$$\Rightarrow xyz\left(1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = 0 \Rightarrow 1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = 0$$

$\Rightarrow x^{-1} + y^{-1} + z^{-1} = -1$

11. ***(a)*** For a skew-symmetric $A = [a_{ij}]$, we have

$a_{ij} = -a_{ji}$ for all $i \neq j$ and $a_{ij} = 0$ for all $i = j$.

Thus, if $A = \begin{bmatrix} 0 & a & 3 \\ 2 & b & -1 \\ c & 1 & 0 \end{bmatrix}$ is skew-symmetric,

then $A_{22} = 0$, $A_{12} = -A_{21}$.

and $A_{31} = -A_{13}$

$\Rightarrow b = 0, a = -2, c = -3$

12. *(b)* We have, $\Delta = \begin{vmatrix} \operatorname{cosec}^2\theta & \cot^2\theta & 1 \\ \cot^2\theta & \operatorname{cosec}^2\theta & -1 \\ 42 & 40 & 2 \end{vmatrix}$

Applying $C_1 \to C_1 - C_2$, we get

$$\Delta = \begin{vmatrix} \operatorname{cosec}^2\theta - \cot^2\theta & \cot^2\theta & 1 \\ \cot^2\theta - \operatorname{cosec}^2\theta & \operatorname{cosec}^2\theta & -1 \\ 42 - 40 & 40 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \cot^2\theta & 1 \\ -1 & \operatorname{cosec}^2\theta & -1 \\ 2 & 40 & 2 \end{vmatrix} = 0 \quad [\because C_1 \text{ and } C_3 \text{ are identical}]$$

13. *(b)* Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin C+\sin^2 C \end{vmatrix}$

$$= \begin{vmatrix} 1 & 1 & 1 \\ \sin A & \sin B & \sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin C+\sin^2 C \end{vmatrix}$$

[applying $R_2 \to R_2 - R_1$]

$$= \begin{vmatrix} 1 & 1 & 1 \\ \sin A & \sin B & \sin C \\ \sin^2 A & \sin^2 B & \sin^2 C \end{vmatrix}$$

[applying $R_3 \to R_3 - R_2$]

$$= \begin{vmatrix} 1 & 0 & 0 \\ \sin A & \sin B - \sin A & \sin C - \sin A \\ \sin^2 A & \sin^2 B - \sin^2 A & \sin^2 C - \sin^2 A \end{vmatrix}$$

[applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$]

$= (\sin B - \sin A)(\sin C - \sin A)$

$$\begin{vmatrix} 1 & 0 & 0 \\ \sin A & 1 & 1 \\ \sin^2 A & \sin B + \sin A & \sin C + \sin A \end{vmatrix}$$

$= (\sin B - \sin A)(\sin C - \sin A)$

$\{\sin C + \sin A - \sin B - \sin A\}$

$= (\sin B - \sin A)(\sin C - \sin A)(\sin C - \sin B)$

Now, $\Delta = 0$

$\Rightarrow (\sin B - \sin A)(\sin C - \sin A)(\sin C - \sin B) = 0$

Either $\sin B = \sin A$ or $\sin C = \sin A$ or $\sin C = \sin B$

$\Rightarrow A = B$ or $C = A$ or $B = C$

$\Rightarrow \Delta ABC$ is isosceles.

14. *(c)* Let $\Delta = \begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = \begin{vmatrix} a^2+2a-3 & 2a-2 & 0 \\ 2a-2 & a-1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$

[applying $R_1 \to R_1 - R_3$, $R_2 \to R_2 - R_3$]

$$= \begin{vmatrix} (a+3)(a-1) & 2(a-1) & 0 \\ 2(a-1) & (a-1) & 0 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^2 \begin{vmatrix} a+3 & 2 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

[taking common $(a-1)$ from R_2 and R_3]

$= (a-1)^2 \cdot 1\,(a+3-4) = (a-1)^3$

15. *(d)* Let Δ be the area of ΔABC.

Then, $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \Rightarrow 2\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

$$\Rightarrow 4\Delta = 2\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & 2 \\ x_2 & y_2 & 2 \\ x_3 & y_3 & 2 \end{vmatrix} \Rightarrow 16\Delta^2 = \begin{vmatrix} x_1 & y_1 & 2 \\ x_2 & y_2 & 2 \\ x_3 & y_3 & 2 \end{vmatrix}^2 \quad \ldots(i)$$

But the area of an equilateral triangle with each side is equal to a is $\frac{\sqrt{3}}{4}a^2$.

$\therefore \quad \Delta = (\sqrt{3}/4)\,a^2 \Rightarrow 16\Delta^2 = 3a^4 \quad \ldots(ii)$

From Eqs. (i) and (ii), we get $\begin{vmatrix} x_1 & y_1 & 2 \\ x_2 & y_2 & 2 \\ x_3 & y_3 & 2 \end{vmatrix}^2 = 3a^4$

16. *(a)* We have $(kA)\cdot\left(\frac{1}{k}A^{-1}\right) = k\cdot\frac{1}{k}(A\cdot A^{-1}) = 1 \times I = I$

$\therefore (kA)$ is inverse of $\frac{1}{k}A^{-1} \Rightarrow (kA)^{-1} = \frac{1}{k}A^{-1}$

17. *(b)* We have, A is a symmetric matrix.

Now, $A^3 = A\cdot A\cdot A \Rightarrow (A^3)^T = (A\cdot A\cdot A)^T$

$= A^T\cdot A^T\cdot A^T$ [by reversal law]

$= A\cdot A\cdot A \quad [\because A^T = A]$

$= A^3$

Hence, A^3 is also a symmetric matrix.

18. *(c)* We have, $\begin{vmatrix} 3 & -2 & \sin 3\theta \\ -7 & 8 & \cos 2\theta \\ -11 & 14 & 2 \end{vmatrix} = 0$

Applying $R_2 \to R_2 + 4R_1$, $R_3 \to R_3 + 7R_1$, we get

$$\begin{vmatrix} 3 & -2 & \sin 3\theta \\ 5 & 0 & \cos 2\theta + 4\sin 3\theta \\ 10 & 0 & 2 + 7\sin 3\theta \end{vmatrix} = 0$$

$\Rightarrow 2[10 + 35\sin 3\theta - 10\cos 2\theta - 40\sin 3\theta] = 0$

$\Rightarrow 10 - 5\sin 3\theta - 10\cos 2\theta = 0$

$\Rightarrow 2 - \sin 3\theta - 2\cos 2\theta = 0$

$\Rightarrow 2 - 3\sin\theta + 4\sin^3\theta - 2 + 4\sin^2\theta = 0$

$\Rightarrow 4\sin^3\theta + 4\sin^2\theta - 3\sin\theta = 0$

$\Rightarrow \sin\theta\,(4\sin^2\theta + 4\sin\theta - 3) = 0$

$\Rightarrow \sin\theta\,(2\sin\theta - 1)(2\sin\theta + 3) = 0$

$\Rightarrow \sin\theta = 0, \frac{1}{2}, -\frac{3}{2}$

But $\sin\theta \neq -\frac{3}{2} \quad [\because \sin\theta \in [-1, 1]]$

$\therefore \sin\theta = 0, \frac{1}{2}$

19. *(c)* We have, $\begin{vmatrix} 1 & -2 & 5 \\ 2 & a & 1 \\ 0 & 4 & 2a \end{vmatrix} = 28$

Applying $R_2 \to R_2 - 2R_1$, we get $\begin{vmatrix} 1 & -2 & 5 \\ 0 & a+4 & -9 \\ 0 & 4 & 2a \end{vmatrix} = 28$

$\Rightarrow 1[(a+4)(2a) - (4)(-9)] = 28 \Rightarrow 2a^2 + 8a + 36 = 28$

$\Rightarrow 2a^2 + 8a + 8 = 0 \Rightarrow a^2 + 4a + 4 = 0$

$\Rightarrow (a+2)^2 = 0 \Rightarrow a = -2, -2$

$\therefore$ Sum is -4.

20. *(d)* We have, $|3AB| = 3^3 \times |A| \times |B|$

$[\because |kA| = k^n|A| \text{ and } |AB| = |A||B|]$

$= 27 \times 5 \times 3 = 405$

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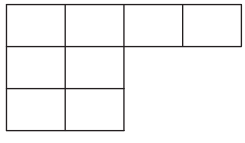
JUST SOLVE & SEND KNOWLEDGE COEFFICIENT QUIZZER (NO. 19)

1. Card are drawn one-by-one at random from a well-shuffled pack of 52 cards until 2 aces are obtained for the first time. If N is the number of cards required to be drawn, then what is the probability of obtaining 2 aces when N is equal to n, where $2 \le n \le 50$

(a) $\frac{(n-2)(50-n)(49-n)}{50 \times 49 \times 16 \times 13}$ (b) $\frac{(n-3)(52-n)(51-n)}{50 \times 49 \times 17 \times 11}$

(c) $\frac{(n-1)(52-n)(51-n)}{50 \times 49 \times 17 \times 13}$ (d) $\frac{(n-2)(49-n)(49-2n)}{49 \times 17 \times 13}$

2. The number of different ways the letters of the word VECTOR can be placed in 8 boxes given below such that no row remains empty is equal to



(a) 26 (b) $26 \times 6!$ (c) $6!$ (d) $2! \times 6!$

3. A normal inclined at an angle of 45° to axis of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is drawn. It meets the major and minor axes at P and Q, respectively. If C is the centre of the ellipse, then area of ΔCPQ is

(a) $\frac{(b^2 - a^2)}{3(a^2 + b^2)}$ sq units (b) $\frac{(2a^2 + b^2)}{2(a^2 - b^2)}$ sq units

(c) $\frac{(a^2 - b^2)^2}{(a^2 + b^2)}$ sq units (d) $\frac{(a^2 - b^2)^2}{2(a^2 + b^2)}$ sq units

4. If $\Delta(x) = \begin{vmatrix} \tan x & \tan(x+h) & \tan(x+2h) \\ \tan(x+2h) & \tan x & \tan(x+h) \\ \tan(x+h) & \tan(x+2h) & \tan x \end{vmatrix}$, then the value of $\lim_{h \to 0} \frac{\sqrt{3}\,\Delta(\pi/3)}{h^2}$ is

(a) 154 (b) 144
(c) 132 (d) 169

5. A window of fixed perimeter (including the base of the arch) is in the form of a rectangle surmounted by a semi-circle. The semi-circle portion is fitted with coloured glass, while the rectangular portion is fitted with clear glass. The clear glass transmits three times as much light per square metre as the coloured glass. Then, the ratio of the sides of the rectangle so that the window transmits the maximum light, is

(a) $\frac{6}{5+\pi}$ (b) $\frac{6}{6+\pi}$ (c) $\frac{1}{3+\pi}$ (d) $\frac{4}{3+\pi}$

6. If the vector $\mathbf{b}, \mathbf{c}, \mathbf{d}$ are not coplanar, then the vector $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) + (\mathbf{a} \times \mathbf{c}) \times (\mathbf{d} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{d}) \times (\mathbf{b} \times \mathbf{c})$ is parallel to

(a) $\mathbf{b}$ (b) $\mathbf{a}$ (c) $\mathbf{c}$ (d) $\mathbf{d}$

7. The value of $\sum_{\alpha,\beta,\gamma} \sin\beta \sin\gamma \cos^2\alpha \sin(\beta - \gamma)$ is

(a) $-\sin(\alpha-\beta)\cos(\gamma-\alpha)\sin(\beta-\gamma)$
(b) $\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta)$
(c) $\cos(\beta-\gamma)\sin(\gamma-\alpha)\sin(\beta-\gamma)$
(d) $-\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta)$

8. If n is a positive integer, then

$$2^n - (n-1)\,2^{n-2} + \frac{(n-2)(n-3)}{2!}2^{n-4} - \frac{(n-3)(n-4)(n-5)}{3!}2^{n-6} + \ldots \text{ is equal to}$$

(a) $n-1$ (b) $2n+1$ (c) $n+1$ (d) n^2+1

9. In a ΔABC, r_A, r_B, r_C are the radii of the circles which touch the incircle and the sides eminating from the vertices A, B, C respectively, then $\sqrt{r_A r_B} + \sqrt{r_B r_C} + \sqrt{r_C r_A}$ is equal to

(a) r (b) $2r$ (c) 0 (d) None of these

10. The equation of the curve passing through the origin in the form $y = f(x)$, which satisfies the differential equation $\frac{dy}{dx} = \sin(10x + 6y)$, is

(a) $y = \frac{1}{2}\tan^{-1}\left[\frac{3}{5}\left\{\tan\left(4x + \tan^{-1}\frac{3}{4}\right)\right\} - \frac{7}{5}\right] - \frac{5x}{3}$

(b) $y = \frac{1}{3}\tan^{-1}\left[\frac{3}{5}\left\{\tan\left(4x + \tan^{-1}\frac{1}{3}\right)\right\} - \frac{3}{4}\right] - \frac{5x}{3}$

(c) $y = \frac{1}{2}\tan^{-1}\left[\frac{4}{5}\left\{\sin\left(4x + \sin^{-1}\frac{3}{4}\right)\right\} - \frac{3}{5}\right] - \frac{5x}{3}$

(d) $y = \frac{1}{3}\tan^{-1}\left[\frac{4}{5}\left\{\tan\left(4x + \tan^{-1}\frac{3}{4}\right)\right\} - \frac{3}{5}\right] - \frac{5x}{3}$

KNOWLEDGE Coefficient

Quizzer (No. 19)

Q. No.	a.	b.	c.	d.	Q. No.	a.	b.	c.	d.
1.	○	○	○	○	6.	○	○	○	○
2.	○	○	○	○	7.	○	○	○	○
3.	○	○	○	○	8.	○	○	○	○
4.	○	○	○	○	9.	○	○	○	○
5.	○	○	○	○	10.	○	○	○	○

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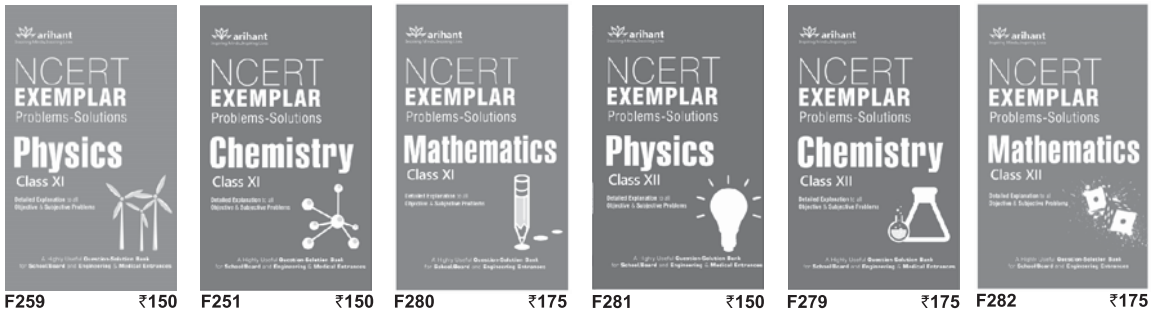
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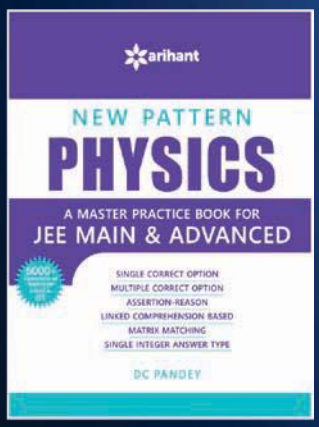

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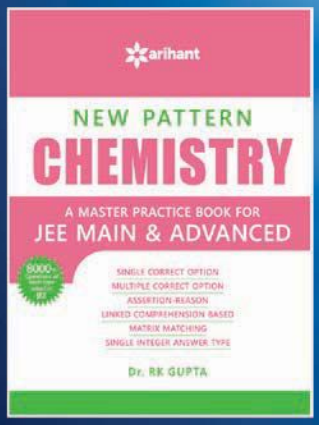

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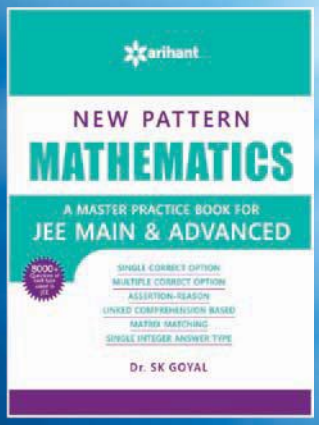

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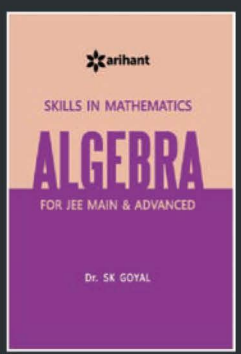

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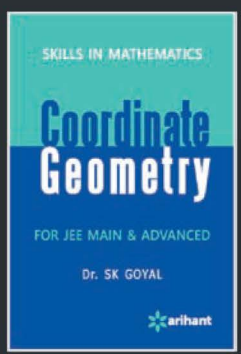

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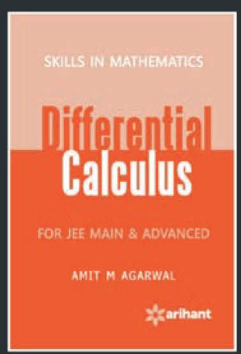

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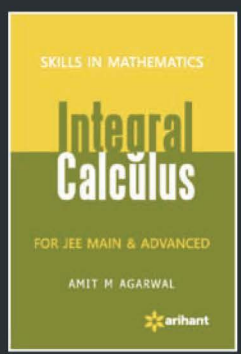

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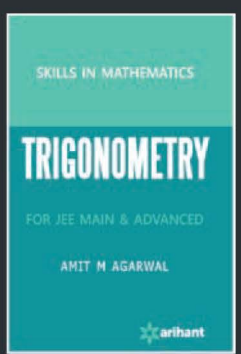

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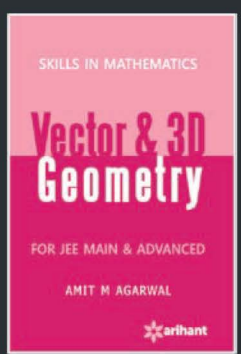

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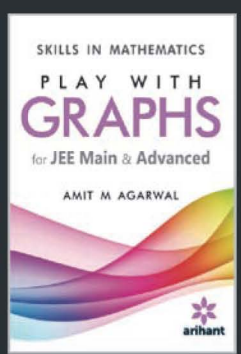

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